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MATHEMATICAL RECREATIONS:

CONTAINING

SOLUTIONS OF MANY VERY DIFFICULT AND IMPORTANT EQUATIONS,

AND OF

SEVERAL USEFUL PROBLEMS

IN

GEOMETRY, SURVEYING AND ASTRONOMY,

TOGETHER WITH A METHOD OF FINDING

THE ROOTS OF EQUATIONS BY PROJECTION.

BY H. N. ROBINSON, A.M.

Author of a Course of Mathematics.

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PREFACE.

When an author comes before the public with a new book, it is customary for him to make some excuse, or to explain some of his reasons for so doing; but in this instance we shall depart from this custom, for we prefer that the leading motive should not be *generally* known. It was not, however, the promptings of vanity, nor the hope of gain: the real motive was far more modest and practical. Some few will perceive the original object designed to be carried out in the work, and to some few I have explained it; and to these it will appear far more perfect and consistent than to others. Without this hint, some would be surprised at the simplicity of several of the problems, and consider them wholly unworthy of a place in a work like this.

This book is not designed for mathematicians *already made*. Such persons require no aid from me. It is for those who have a desire to *become* mathematicians, and for those who would render their knowledge a little more practical than can be deduced from the ordinary class-books in this country.

The lovers of the mathematics will perhaps be pleased with the manner of finding the roots of equations of the second and third degrees, which is here carried out more fully and practically than in any work that has come under the author's inspection. The subject is touched upon in Hutton's Mathematics as formerly published; but I claim to have elucidated, modernized, and rendered it practically useful, so far as respects *real* roots; and there is reason to hope

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that certain abstract principles which are to be found in a recent work by Mr. John Paterson of Albany, will ere long enable us satisfactorily to geometrise *imaginary* roots.

In discussing this subject, I arrived at equations of the fourth degree prior to those of the third. This course is doubtless open to criticism, and may perhaps be condemned by some, as deviating too widely from the usual mode of communicating instruction by proceeding from the simpler to the more complicated cases; but we would caution the censorious, and remind them that different roads, from different points of the compass, may yet lead to the same citadel.

I have not referred to the differential and integral calculus; fearing, if such were the case, that many would lay aside the book, to whom it would otherwise be highly useful. Some useful and practical problems from these sciences may nevertheless be added at a future day, if circumstances should warrant such a course.

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MATHEMATICAL RECREATIONS.

EQUATIONS.

1. Given $(x-\frac{1}{x})^{\frac{1}{2}}+(1-\frac{1}{x})^{\frac{1}{2}}=x$: to find the values of x.

Put $(x-\frac{1}{x})^{\frac{1}{2}}=P$, and $(1-\frac{1}{x})^{\frac{1}{2}}=Q$; ----------[1]

then the given equation becomes $P+Q=x$. ----------[2]

Multiply [2] by $P-Q$; then we have

$$P^2-Q^2=(P-Q)x. \text{ ----------[3]}$$

From [1], we find $P^2-Q^2=(x-1)$;

therefore $$x-1=(P-Q)x,$$

or $$1-\frac{1}{x}=P-Q. \text{ ----------[4]}$$

By adding [2] and [4], we find

$$1-\frac{1}{x}+x=2P;$$

that is, $$1+P^2=2P.$$

By transposition, $1-2P+P^2=0$, or $1-P=0$, or $P=1$.

Therefore, by the first of equations [1], we perceive that

$$x-\frac{1}{x}=1, \text{ or } x^2-x=1:$$

Hence $$x=\tfrac{1}{2}(1\pm\sqrt{5}).$$

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2. Given $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$: to find the values of x.

We observe that this equation is in the same form as the preceding, and would be identical if we changed x^2 to x, a^4 to 1. Therefore the value of x^2 in this equation will be of the same form as the value of x in the first example, except it will contain the factor a^2, because the square root has been once extracted:

That is, $$x^2=\frac{a^2}{2}(1\pm\sqrt{5}),$$

or $$x=\mp a\left(\frac{1\pm\sqrt{5}}{2}\right)^{\frac{1}{2}}.$$

But this conclusion is too summary to satisfy the young algebraist; therefore it is proper to take some of the intermediate steps.

Put $$\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=P, \qquad \left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=Q; \text{ ---------- [1]}$$

then the given equation becomes

$$P+Q=\frac{x^2}{a}. \text{ ---------- [2]}$$

Multiply [2] by $(P-Q)$; then we have

$$P^2-Q^2=\frac{x^2}{a}(P-Q).$$

But the value of (P^2-Q^2) drawn from [1], is (x^2-a^2);

therefore $$\frac{x^2}{a}(P-Q)=x^2-a^2,$$

or $$P-Q=a-\frac{a^3}{x^2}. \text{ ---------- [3]}$$

By adding equations [2] and [3], we find

$$2P=a-\frac{a^3}{x^2}+\frac{x^2}{a}. \text{ ---------- [4]}$$

Multiply this equation by a; then

$$2aP=a^2-\frac{a^4}{x^2}+x^2,$$

that is, $$2aP=a^2+P^2,$$

or $$0=a^2-2aP+P^2.$$

Square root, $$0=a-P, \quad \text{or } P=a,$$

$$P^2=a^2.$$

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From the first of equations [1], we find

$$x^2 - \frac{a^4}{x^2} = a^2.$$

From this equation, we find $x = \mp a\left(\frac{1 \pm \sqrt{5}}{2}\right)^{\frac{1}{2}}$.

3. Given $x^4\left(1+\frac{1}{3x}\right)^2 - (3x^2+x) = 70$: to find the values of x.

Observe that $(3x^2+x) = 3x^2\left(1+\frac{1}{3x}\right)$. Put $\left(1+\frac{1}{3x}\right) = y$;

then the given equation becomes $x^4y^2 - 3x^2y = 70$.

Completing the square, $x^4y^2 - 3x^2y + \frac{9}{4} = \frac{289}{4}$;

$$x^2y - \frac{3}{2} = \pm\frac{17}{2};$$

$$x^2y = 10, \text{ or } -7;$$

that is, $x^2 + \frac{x}{3} = 10, \text{ or } -7.$

Whence $x = 3$, or $-\frac{10}{3}$, or $\frac{1}{6}(1 \pm \sqrt{-251})$.

4. Given $\dfrac{(x+y)+\sqrt{x^2-y^2}}{(x+y)-\sqrt{x^2-y^2}} = \dfrac{9(x+y)}{8y}$,

and $(x^2+y)^2+(x-y) = 2x(x^2+y)+506$,

to find the values of x and y.

Multiply the fraction, numerator and denominator, which forms the first member of the first equation, by $(x+y)+\sqrt{x^2-y^2}$;

then we shall have $\dfrac{((x+y)+\sqrt{x^2-y^2})^2}{2xy+2y^2} = \dfrac{9(x+y)}{8y}$.

Divide both denominators by $2y$; then

$$\frac{(x+y)+\sqrt{x^2-y^2})^2}{x+y} = \frac{9(x+y)}{4}.$$

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Multiply by $(x+y)$, and then extract the square root of both members; then $x+y+\sqrt{x^2-y^2} = \frac{3}{2}(x+y)$: ----------[a]

Hence $2\sqrt{x^2-y^2} = (x+y)$:

Squaring, $4(x^2-y^2) = (x+y)^2$.

Divide by $(x+y)$; then we have

$$4(x-y) = x+y, \quad \text{or } y = \frac{3x}{5}.$$

In the second equation, put $x^2+y = P$; then

$$P^2+x-y = 2xP+506,$$

or $$P^2-2xP = -x+y+506.$$

Add x^2 to both members of this equation; then

$$P^2-2xP+x^2 = x^2+y-x+506,$$

or $$(P-x)^2 = x^2+y-x+506,$$

or $$(x^2+y-x)^2 = (x^2+y-x)+506;$$

that is, $$Q^2-Q = 506,$$

or $$Q = 23 \text{ or } -22;$$

that is, $$x^2+y-x = 23 \text{ or } -22,$$

or $$x^2+\tfrac{3}{5}x-x = 23 \text{ or } -22,$$

or $$x^2-\tfrac{2}{5}x = 23 \text{ or } -22 :$$

Hence $x = 5$ or $-\frac{23}{5}$, $\quad y = 3$ or $-\frac{69}{25}$.

To obtain all the roots in these compound equations, we must notice the double sign, and use all the results that arise from extracting square roots; and we shall find a greater or less number of roots, according to the number of times that we extract the square root.

5. Find the values of x from the following equation :

$$2x^{\frac{3}{2}}(x^3+a^3)^{\frac{1}{2}} = 2x^2(x+2a)+a^2(x-a).$$

Put $a = nx$; then

$$2x^3(1+n^3)^{\frac{1}{2}} = 2x^2(x+2nx)+n^2x^2(x-nx).$$

Divide both members by x^3; then we have

$$2(1+n^3)^{\frac{1}{2}} = 2(1+2n)+n^2(1-n)$$
$$= 2+4n+n^2-n^3.$$

Add 1 to both members, and transpose $-n^3$; then we have

$$(1+n^3)+2(1+n^3)^{\frac{1}{2}} = 3+4n+n^2.$$

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Add 1 to both members, to complete the squares; then extract the square root, and we shall have

$$(1+n^3)^{\frac{1}{2}}+1 = \pm(2+n).$$

Taking the plus sign, dropping unity from both sides, and squaring, gives $1+n^3 = (1+n)^2$. ----------[A]

By division, $1-n+n^2 = 1+n$, or $n = 2$.

Eq. [A] can be put in this form :

$$\overline{(1-n+n^2)}-\overline{(1+n)}\times(1+n) = 0.$$

This equation can be verified by making either factor equal zero : the first factor gives $n = 2$; the second, $n = -1$.

Whence, $x = \frac{1}{2}a$ or $-a$.

---

6. Given $\sqrt{x}-\frac{8}{x} = \frac{7}{\sqrt{x}-2}$, to find the values of x.

Put $\sqrt{x} = y$; then $y-\frac{8}{y^2} = \frac{7}{y-2}$;

$$y^4-2y^3-8y+16 = 7y^2;$$

$$y^4-2y^3-7y^2-8y+16 = 0.$$

Extract the square root, or rather attempt the extraction :

$$y^4-2y^3-7y^2-8y+16 = 0\,(\,y^2-y$$

$$2y^2-y\overline{)\;y^4\;-2y^3-7y^2}$$

$$\underline{\quad -2y^3+\;y^2}$$

$$-8y^2-8y+16$$

Now the equation may be put in the following form :

$$(y^2-y)^2-8y^2-8y+16 = 0.$$

If $-8y^2$ were $+8y^2$, the equation would be in a quadratic form; but we can cause $+8y^2$ to appear in the first member of the equation, by adding $16y^2$ to both members : then we shall have

$$(y^2-y)^2+8y^2-8y+16 = 16y^2,$$

or $$(y^2-y)^2+8(y^2-y)+16 = 16y^2.$$

By evolution, $y^2-y+4 = \pm 4y$;

therefore $y^2-5y = -4$, and $y^2+3y = -4$;

or $y = 4$ or 1, and $y = \frac{1}{2}(\pm\sqrt{-7}-3)$.

But $\sqrt{x} = y$; $\sqrt{x} = 4$ or 1, $x = 16$ or 1.

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7. Given $xy = 125x+300y$, and $y^2-x^2 = 90000$, to find x and y.

A direct solution of this problem, and without taking especial care, will produce a resulting equation containing very high numbers : to avoid this inconvenience, we must resort to artifice.

Observe that $5\times25=125$, $12\times25=300$, $(300)^2=90000$.

Put $a = 25$, $b = 12$;

then $xy = 5ax+aby$, $y^2-x^2 = a^2b^2$; ----------[1]

$$x = \frac{aby}{y-5a}; \quad \text{----------[2]}$$

$$x^2 = \frac{a^2b^2y^2}{y^2-10ay+25a^2}.$$

Substitute this value of x^2 in [1]; then we have

$$y^2-\frac{a^2b^2y^2}{y^2-10ay+25a^2} = a^2b^2;$$

$$y^4-10ay^3+25a^2y^2-a^2b^2y^2 = a^2b^2y^2-10a^3b^2y+25a^4b^2;$$

$$y^4-10ay^3+(25a^2-2a^2b^2)y^2+10a^3b^2y = 25a^4b^2.$$

Decompose the first member of this last equation into factors, by attempting to extract the square root, thus :

$$y^4-10ay^3+25a^2y^2-2a^2b^2y^2+10a^3b^2y(y^2-5ay$$

$$2y^2-5ay)\ \underline{y^4\ -10ay^3+25a^2y^2}$$
$$\underline{-10ay^3+25a^2y^2}$$
$$-2a^2b^2(y^2-5ay)$$

This result shows that the last equation may be written in the following form :

$$(y^2-5ay)^2-2a^2b^2(y^2-5ay) = 25a^4b^2.$$

Put $y^2-5ay = Q$; then we have the quadratic

$$Q^2-2a^2b^2Q = 25a^4b^2;$$

$$Q^2-2a^2b^2Q+a^4b^4 = 25a^4b^2+a^4b^4$$
$$= a^4b^2(25+b^2)^*$$
$$= a^4b^2(169).$$

By evolution, $Q-a^2b^2 = 13a^2b$;

$$Q = 13a^2b+a^2b^2 = a^2b(13+b) = 25a^2b = a^3b.$$

* Observe that $b^2 = 144$; therefore $(25+b^2)=169$: Also $a = 25$; therefore $25a^2b = a^3b$.

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Hence
$$\begin{aligned} y^2-5ay &= a^3b; \\ 4y^2-20ay+25a^2 &= 4a^3b+25a^2 \\ &= 4a^3b+a^3 \\ &= a^3(4b+1) \\ &= 49a^3 = 49a^2.25; \\ 2y-5a &= 7.a.5; \\ 2y &= 40a; \\ y &= 20a = 500. \end{aligned}$$

This value of y, substituted in equation [2], gives

$$x = \frac{ab.20a}{15a} = \frac{25.12.4}{3} = 16.25.$$

8. Given $\left\{\begin{array}{r} x+xy^5 = 99, \\ xy+xy^2+xy^3+xy^4 = 90, \end{array}\right\}$ to find the values of x and y.

Divide one equation by the other, and we have

$$\frac{1+y^5}{y+y^2+y^3+y^4} = \frac{11}{10}.$$

Reducing the first member by dividing numerator and denominator by $(1+y)$, and

$$\frac{1-y+y^2-y^3+y^4}{y+y^3} = \frac{11}{10}.$$

Clearing of fractions,

$$\begin{aligned} 10-10y+10y^2-10y^3+10y^4 &= 11y+11y^3; \\ 10(1+y^2+y^4) &= 21(y+y^3). \end{aligned}$$

Divide both members by y^2; then we have

$$10(\frac{1}{y^2}+1+y^2) = 21(\frac{1}{y}+y).$$

Now put $(\frac{1}{y}+y) = P$; then $\frac{1}{y^2}+2+y^2 = P^2$.

Whence $10(P^2-1) = 21P$,

or $P^2-\frac{21}{10}P = 1, \qquad P = \frac{5}{2}$.

Hence
$$y+\frac{1}{y} = \frac{5}{2};$$
$$y^2-\tfrac{5}{2}y = -1: \qquad \text{whence } y = 2.$$

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Put this value of y in the first equation,

$$x+xy^5 = 33x = 99;$$
$$x = 3.$$

These equations obviously result from the following problem :

"There are six numbers in geometrical progression : the sum of the first and sixth is 99, and the sum of the four means is 90; what are the numbers?" Ans. 3, 6, 12, 24, 48, 96.

The numbers may be denoted by $\frac{x^3}{y^2}$, $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$ and $\frac{y^3}{x^2}$.

(See ROBINSON'S *Algebra*, page 198, art. 123, and art. 112.)

We call attention to this, to show that equations may sometimes be changed to others having no visible connection.

9. Find two numbers, whose product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

In other words, find the values of x and y from the following equations :

$$xy = x^2-y^2, \quad \text{----------[1]}$$
$$x^2+y^2 = x^3-y^3. \quad \text{----------[2]}$$

Put $x = vy$; then [1] and [2] become

$$vy^2 = v^2y^2-y^2, \quad \text{or} \quad v = v^2-1, \quad \text{------[3]}$$
$$v^2y^2+y^2 = v^3y^3-y^3, \quad \text{or} \quad v^2+1 = (v^3-1)y. \quad \text{------[4]}$$

From [3], $1+v = v^2$; whence $2+v = v^2+1$. ------[5]

Multiply the first of equations [5] by v, and we have

$$v+v^2 = v^3;$$

that is, $2v+1 = v^3$, or $2v = v^3-1$. ----------[6]

By the aid of [5] and [6], we perceive that [4] reduces to this form :

$$2+v = 2vy,$$

or

$$y = \frac{2+v}{2v}. \quad \text{----------[7]}$$

Equation [3], resolved as a quadratic, gives

$$2v = 1\pm\sqrt{5}, \quad \text{and} \quad 2+v = \tfrac{1}{2}(\pm\sqrt{5}+5).$$

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These values, put in the second member of equation [7], give

$$y = \frac{\frac{1}{2}(\pm\sqrt{5}+5)}{(1\mp\sqrt{5})} = \pm\tfrac{1}{2}\sqrt{5} \text{ by actual division.}$$

But $x = vy = \frac{1}{2}(1\pm\sqrt{5})\times\pm\frac{1}{2}\sqrt{5} = \frac{1}{4}(\pm\sqrt{5}\pm 5)$.

10. Given $\left\{\begin{array}{c}(x-2)y-\sqrt{xy}(y^2-1) = 2y^2-x,\\ xy = \dfrac{4\sqrt{xy}-48}{xy-18},\end{array}\right\}$

to find the rational values of x and y.

Ans. $x = 8$, or -12;
$y = 2$, or $-\frac{4}{3}$.

Transpose $-2y$ and $-x$ in the first equation, and we have

$$xy+x-\sqrt{xy}(y^2-1) = 2y^2+2y.$$

Divide both members by $(y+1)$; then

$$x-\sqrt{xy}(y-1) = 2y. \quad \text{----------[a]}$$

This is as far as we can reduce this equation, without the aid of the other equation; therefore clear the second equation of fractions, and we have

$$x^2y^2-18xy = 4\sqrt{xy}-48.$$

Add 49 to both members; then

$$x^2y^2-18xy+49 = 4\sqrt{xy}+1.$$

Add $4xy$ to both members, to make squares of them; then

$$x^2y^2-14xy+49 = 4xy+4\sqrt{xy}+1.$$

By evolution, $xy-7 = 2\sqrt{xy}+1$, ----------[n]

or $xy-2\sqrt{xy} = 8$;

$xy-2\sqrt{xy}+1 = 9$;

$\sqrt{xy}-1 = \pm 3$;

$\sqrt{xy} = 4$ or -2. ----------[m]

Put these values of $\sqrt{xy}$ in equation [a], and we have

$x-4(y-1) = 2y$, ----------[b]

or $x+2(y-1) = 2y$. ----------[c]

From [b], $x-6y+4 = 0$; ----------[d]

from [c], $x = 2$.

(Math. Recreations.)

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From equations [d] and [m], we find $y = 2$ or $-\frac{4}{3}$.

These values of y, put in equation [m], give $x = 8$ or -12.

By giving the double sign to the second member of [n], other values can be found which may or may not be imaginary. Observe that we divided one of the equations by $(y+1)$; therefore $(y+1)$ may be put equal to zero, and -1 must be one root of that equation.

---

11. Find some of the values of x and y from the following equations :

$$3x - x\sqrt{\frac{5x^2}{4} - 2y + 8} = 2 - y, \quad \text{----------[1]}$$

$$\frac{\sqrt{x+y}}{2x} - \frac{3x}{4} = \frac{2x-3}{\sqrt{x+y}} - \frac{3y}{2x}. \quad \text{----------[2]}$$

Double equation [1], then add $(\frac{5x^2}{4}+4)$ to both members; then we have

$$\frac{5x^2}{4} + 4 + 6x - 2x\sqrt{\frac{5x^2}{4} - 2y + 8} = (8 - 2y + \frac{5x^2}{4}).$$

Put $\sqrt{\frac{5x^2}{4} - 2y + 8} = P$, and transpose; then

$$\frac{5x^2}{4} + 6x + 4 = P^2 + 2xP.$$

Now add x^2 to both members, and

$$\frac{9x^2}{4} + 6x + 4 = P^2 + 2xP + x^2.$$

By evolution, $$\frac{3}{2}x + 2 = P + x,$$

or $$\frac{x}{2} + 2 = P.$$

By squaring, $$\frac{x^2}{4} + 2x + 4 = P^2 = \frac{5x^2}{4} - 2y + 8,$$

or $$2x = x^2 - 2y + 4. \quad \text{----------[a]}$$

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Multiply eq. [2] by $4x$, and transpose $-3x^2$; then we have

$$2\sqrt{x+y} = \frac{(2x-3)4x}{\sqrt{x+y}} + 3x^2 - 6y.$$

From [a], we find $3x^2-6y = 6x-12$; therefore

$$2\sqrt{x+y} = \frac{(2x-3)4x}{\sqrt{x+y}} + 6x - 12.$$

Dividing by 2, and afterwards multiplying by $\sqrt{x+y}$, gives

$$x+y = (2x-3)2x+(3x-6)\sqrt{x+y}.$$

From eq. [a], we find $x+y = (\frac{x^2}{2}+2)$ and $\sqrt{x+y} = \sqrt{\frac{x^2}{2}+2}$;

therefore $(\frac{x^2}{2}+2) = 4x^2-6x+(3x-6)\sqrt{\frac{x^2}{2}+2}$.

Put $\sqrt{\frac{x^2}{2}+2} = Q$; then we have

$$Q^2-(3x-6)Q = 4x^2-6x.$$

Add $(\frac{3x}{2}-3)^2$ to both members, to complete the squares; then

$$Q^2-(3x-6)Q+(\frac{3x}{2}-3)^2 = \frac{25x^2}{4}-15x+9.$$

By evolution, $Q+\frac{3}{2}x-3 = \frac{5}{2}x-3$; ..... [b]

whence $Q = x$;

that is, $\sqrt{\frac{x^2}{2}+2} = x$, or $\frac{x^2}{2}+2 = x^2$, or $x = 2$.

This value of x, put in [a], gives $y = 2$.

By taking the double sign to the second member of equation [b], we can find other values of x and y.

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12. Find some of the values of x and y from the following equations :

$$5-2\sqrt{y+2} = \frac{9x^2}{64}-(\sqrt{x}-3\sqrt{y})^2, \text{ ----------[1]}$$

$$\frac{7}{y}-10\sqrt{\frac{x}{y}} = x-16. \text{ ----------[2]}$$

$$7-10\sqrt{xy} = xy-16y.$$

By transposition, $7+16y = xy+10\sqrt{xy}$.

Add 25, $32+16y = xy+10\sqrt{xy}+25$.

By evolution, $4\sqrt{2+y} = \sqrt{xy}+5$. ----------[a]

By transposing $(\sqrt{x}-3\sqrt{y})^2$ in eq.[1], and expanding, we have

$$5-2\sqrt{y+2}+x-6\sqrt{xy}+9y = \frac{9x^2}{64}. \text{ --------[b]}$$

Equation [a] shows us that $6\sqrt{xy} = 24\sqrt{2+y}-30$.

Substituting this value of $6\sqrt{xy}$ in [b], we have

$$5-2\sqrt{y+2}+x-24\sqrt{y+2}+30+9y = \frac{9x^2}{64}.$$

Uniting and transposing, gives

$$35+9y-26\sqrt{y+2} = \frac{9x^2}{64}-x.$$

Subtract 17 from both members; then we may write the equation thus : $9(2+y)-26\sqrt{2+y} = \frac{9x^2}{64}-x-17$.

Put $\sqrt{2+y} = P$, and complete the square of the first member by Art. 99 in Robinson's Algebra; then we shall have

$$9P^2-26P+\frac{169}{9} = \frac{9x^2}{64}-x+\frac{16}{9}.$$

By evolution, $3P-\frac{13}{3} = \frac{3x}{8}-\frac{4}{3}$;

whence $P = \frac{x}{8}+1$, ----------[c]

or $4P = \frac{x}{2}+4$. ----------[d]

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Comparing equations [a] and [d], we perceive that

$$\frac{x}{2}+4 = \sqrt{xy}+5,$$

or $$\frac{x}{2}-1 = \sqrt{xy}.$$

By squaring, $$\frac{x^2}{4}-x+1 = xy,$$

or $$\frac{x}{4}-1+\frac{1}{x} = y. \quad \text{----------[n]}$$

Equation [c] is $$\frac{x}{8}+1 = \sqrt{2+y}.$$

By squaring, $$\frac{x^2}{64}+\frac{x}{4}+1 = 2+y,$$

or $$\frac{x^2}{64}+\frac{x}{4}-1 = y. \quad \text{----------[m]}$$

Equations [n] and [m] give

$$\frac{x^2}{64}+\frac{x}{4}-1 = \frac{x}{4}-1+\frac{1}{x},$$

or $$\frac{x^2}{64} = \frac{1}{x}, \qquad \text{or } x = 4, \quad y = \tfrac{1}{4}.$$

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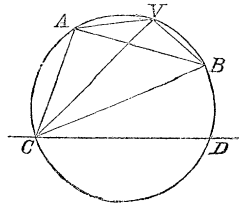
GEOMETRICAL PROBLEMS.

1. *From two given points on the same side of a line given in position, it is required to draw two straight lines which shall contain a given angle, and be terminated in that line.*



Let A, B be the two given points, and CD the line given in position.

Join AB; and from the point A, and on the opposite side of AB from CD, make the angle ABV equal to *half* the given angle; and from the point B, make the angle BAV also equal to half the given angle, thus forming the isosceles triangle AVB. About the triangle AVB describe a circle, cutting the line given in position in the two points C and D. Join AC and CB, or AD and BD, and they will be the lines required.

DEMONSTRATION.

Join CV. The angle VCB = BAV, because each is measured by half the same arc VB. Also the angle ACV = ABV, because each is measured by half the same arc AV.

Now by addition the angle ACB = BAV+ABV = the given angle. Q. E. D.

Remark. The given angle can not be an angle of *any* magnitude : it must be sufficiently small to permit the circle to cut the line which is given in position. When the greatest possible angle is required, the circle will pass through the two points A and B, and touch the line CD.

This observation suggests the following problem :

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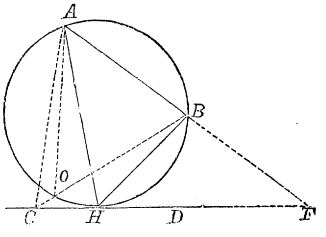
2. *From two given points on the same side of a line given in position, it is required to draw two lines that shall meet in that line, at that point where they would form the* GREATEST POSSIBLE *angle.*



Let A, B be the two given points, and CD the line given in position. Produce AB and CD until they meet at F. Take FH of such a magnitude as to make

$$(FH)^2 = FA \times FB.$$

Through the three points A, B and H, describe a circle; the line FH will touch that circle at the point H. Join AH, BH, and the angle AHB is the angle required; and it is the greatest angle that can be made by lines drawn from A and B, to terminate in the line CD.

DEMONSTRATION.

Take any other point than H, as C on the line CD, and join BC and AC; these lines will cut the circumference of the circle, because CD touches the circle at H.

BC cuts the circle in O. Join AO.

Now the angle AOB = AHB, because each is measured by half the same arc AB; but AOB is greater than ACB, because it is the exterior angle of the triangle AOC; that is, AHB is greater than ACB.

In the same manner we could prove that the angle AHB would be greater than ACB, if the point C were in any other position on the line CD, except H; therefore AHB is the greatest possible angle required.

Remark. In case AB and CD would not meet, AB would be parallel to CD; and then we would find the point H by making the triangle AHB isosceles, having AB for its base, and its vertex terminated in the line CD.

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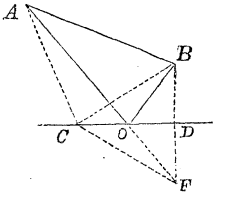
3. *From two given points on the same side of a line given in position, it is required to draw two straight lines that shall meet in that line, and make the* LEAST POSSIBLE *sum.*



Let A, B be the two given points, and CD the line given in position.

From either point as B, let fall the perpendicular BD on the line CD, and produce it to F, making BD = DF. Join AF, and from O the intersection of CD join OB. AO+OB is less than the sum of any other two lines drawn from the points A and B, and terminated in the line CD.

DEMONSTRATION.

Because BD = DF; OD common to the two rightangled triangles BDO, ODF; OB = OF : therefore AO+OB = AF.

Join AC, CB, CF. AC+CB = AC+CF, because CB=CF; but (AC+CF) are together greater than the straight line AF; that is, (AC+CB) are together greater than (AO+OB).

Now let C be at any other point in the line CD, except O, and still we shall have (AC+CB) greater than (AO+OB) : hence (AO+OB) is the least possible.

Remark. We may observe that the angle BOD = AOC. Consider CD a plane mirror, and BO a ray of light passing from B to O : it would then be reflected in the line OA, as we well know by observation; therefore when light passes from one point, and is reflected to another, it always passes through the shortest distance in space.

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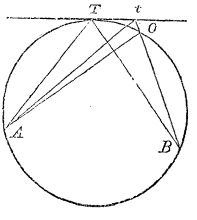
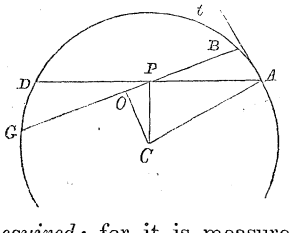
4. *If, from any two points in the circumference of a circle, there be drawn two straight lines to a point in a tangent to that circle, they will make the greatest angle when drawn to the point of contact.*



Let ATB be the circle; Tt, a tangent at the point T; and A, B, any two points in the circumference. Draw AT, BT, At, Bt, and join AO.

The angle ATB = AOB, because they stand on the same arc AB. But the angle AOB is greater than the angle AtB, because it is the exterior angle of the triangle AtO. Therefore the angle ATB, which is equal to AOB, is greater than the angle AtB; and the same would be true if t were at any other point on the tangent except T.

5. *From any point within a given circle, to draw a straight line, which shall make with the circumference an angle less than any angle made by any other line drawn from that point.*



Let P be any point in the circle. Draw PC to the centre; and through P draw APD at right angles to PC, and draw At a tangent for the point A. The angle PAt is the angle which the line PA makes with the circumference, and it is the *angle required;* for it is measured by half the arc ABD, and the arc ABD is *less than any other arc* that can be cut off by a line passing through P.

To show this, draw any other chord through P, as BPG, and from the centre let CO fall perpendicularly upon it. CO is less than CP; therefore the chord BG is greater than the chord AD, and consequently the arc BDG is greater than the arc ABD; and as the angle that PB makes with the circumference is measured

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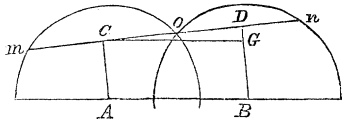
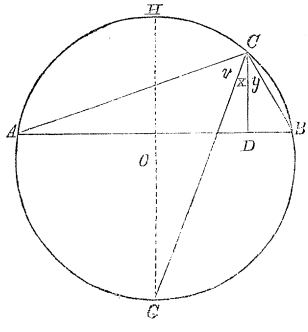
by half the arc BDG, therefore it *is greater* than the angle which PA makes with the circumference; and as PB is any line drawn through P other than PA, therefore PA makes with the circumference a less angle than any other line drawn through P.

6. *If two circles cut each other, the greatest line that can be drawn through the point of intersection, is that which is parallel to the line joining the centres.*



Let two circles cut each other at the point O, and through O draw any line nm. From the centres A and B, let fall perpendiculars AC, BD on the line mn. Through C draw CG parallel to AB; thus forming the parallelogram ACGB, and the rightangled triangle CDG rightangled at D. Therefore CG (=AB) is greater than CD, or CD is *less* than AB. Now if mn were parallel to AB, CD would be equal to AB, and greater than it now is in an inclined position. But CD is the half of mn; therefore the greatest line than can be drawn through O, is the one parallel to AB, *and it will be equal to twice* AB.

7. *The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.*



Let ABC be a triangle, and circumscribe it by a circle.

Through the centre O, draw GOH perpendicular to the base AB : it will then bisect AB and the arc AB.

Join CG, and this line will bisect the angle ACB, because the arc AG = GB. Let fall CD per-

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pendicular to AB; then CD will be parallel to GH, and the angle HGC = GCD.

The angle ABC is measured by $\frac{1}{2}$ the arcs AH+HC;

the angle BAC is measured by $\frac{1}{2}$ the arcs AH—HC: therefore, their difference is measured by $\frac{1}{2}$ the arc 2 HC, or by the arc HC. But the angle HGC, or its equal GCD, is measured by one-half the arc HC; therefore the difference of the angles at the base is double of the angle GCD.

OTHERWISE.

(See the notation in the figure.)

We have $x+y = v$;

whence, $y-v = -x$. ..... [1]

$y+B = 90°$;

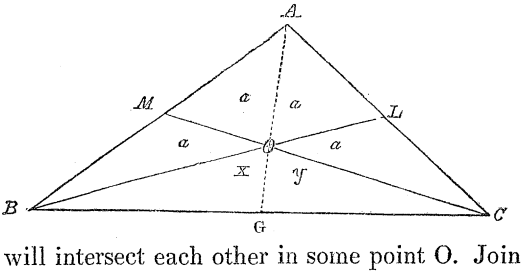
$v+x+A = 90°$.

By subtraction, $(y-v)-x+B-A = 0$;

that is, $-x-x+(B-A) = 0$,

or $(B-A) = 2x$. Q. E. D.

8. *If, from the three angles of any triangle, lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.*



Let ABC be a triangle; and from the angle B draw BL to the middle of AC, and CM to the middle of AB: these lines will intersect each other in some point O. Join AO, and produce it to G. *We are now to prove that* BG = GC.

Triangles which have the *same vertex*, are to one another as their bases. Now the triangle ABL is half the triangle ABC, because AL is the half of AC, and the two triangles have the

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same vertex B. For a like reason, the triangle AMC is also half the triangle ABC; therefore the triangle ABL = AMC.

From these two triangles, take away the common space AMOL, and the remainders LOC = MOB. Designate each of these triangles by a. The triangles LOC and AOL are equal, because they have the same vertex O and equal bases AL, LC; therefore AOL is also a. For a like reason, the triangle AMO is a. Designate the triangle BOG by x, and GOC by y; and if we prove $x = y$, it then follows that BG = GC.

Now as AOG is a right line, the triangle AOB is to BOG, as AO to OG; that is,

$$2a : x = \text{AO} : \text{OG}.$$

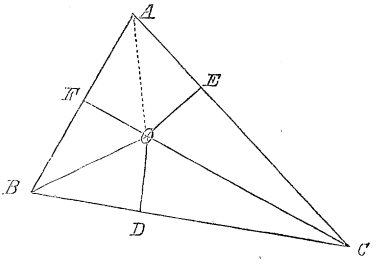
Also, $2a : y = \text{AO} : \text{OG}$;

hence $2a : x = 2a : y$,

or $x = y$.

Hence, BG = GC, which shows that the line which joins AO produced will bisect the base.

9. *The three straight lines which bisect the three angles of a triangle, meet in the same point.*



Let ABC be the triangle. Bisect the angles B and C by the lines BO and CO; and from the point O let drop the perpendiculars OD, OF, OE, upon the sides.

Join AO by the dotted line. *We are now to prove that the angle* FAO = EAO.

The two triangles BFO, BDO are equal, because they have the common hypothenuse BO, and the angle FBO = DBO; therefore FO=DO. In the same manner we prove that DO=OE; whence FO=OE. Now the two rightangled triangles AFO, AEO have the same hypothenuse AO, and one side FO equal to one side OE; therefore the angle FAO = EAO, which proves the theorem.

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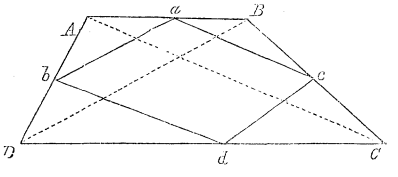
Remark. It is obvious that O is the centre, and OD, OF, OE radii of the inscribed circle.

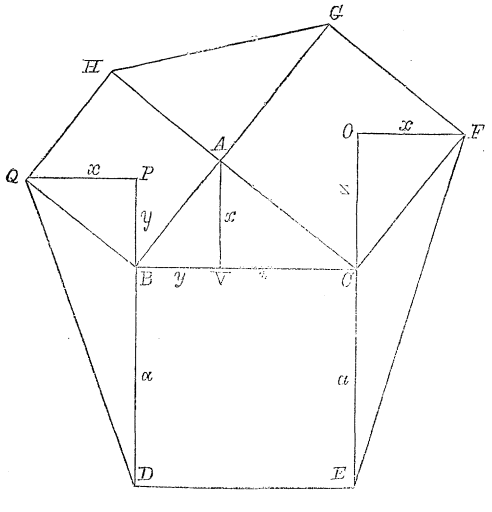
10. *The figure formed by joining the points of bisection of the sides of a trapezium, is a parallelogram.*



Let ABCD be a trapezium. Draw the dotted diagonals AC and BD. Bisect AB in a, BC in c, and join ac.

Now the sides of the triangle ABC are cut proportionally; therefore ac is parallel to AC, and equal to half of it. In the same manner we prove that bd is also parallel to AC, and equal to half of it; therefore ac and bd are parallel and equal, and in the same manner we prove that ab and dc are parallel and equal : consequently abdc is a parallelogram.

11. *If squares be described on the three sides of a rightangled triangle, and the extremities of the adjacent sides be joined, the triangles so formed are equal in* AREA *to the given triangle, and to each other.*



Let ABC be the given rightangled triangle, and construct the figure as in the margin.

We must now prove that

$$QP = AV.$$

The angle ABQ is a right angle, and equal to PBV. From each of these equals, take away the common angle

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PBA, and QBP = ABV. The angles at P and V are right angles, and the sides opposite to them QB and AB are equal, being sides of the same square; therefore the two triangles PQB and ABV are in all respects equal, and QP = AV.

In the same manner, we can prove OF = AV.

Now observe that BD=CE=BC. Put BC = a and AV = x; then QP = x, OF = x.

Observe that ax is double of the area of the triangle ABC, and that BD×OP = the double of the triangle OBD; but BD×OP = ax; also CD×OF = ax; whence the triangles ABC, QDB, FCE are all equal in area. The triangle HAG is obviously equal to ABC; therefore each of the exterior triangles is in area equal to the original triangle ABC.

12. *Taking the last figure, we can prove that the sum of the squares of* (QD) *and* (FE) *is equal to five times the square of the hypothenuse* (BC).

In the rightangled triangles QPD, OFE, we have

$$x^2+(y+a)^2 = (\mathrm{QD})^2,$$

$$x^2+(z+a)^2 = (\mathrm{FE})^2.$$

Add, $2x^2+y^2+z^2+2a(y+z)+2a^2 = (\mathrm{OD})^2+(\mathrm{FE})^2$;

that is, $(x^2+y^2)+(x^2+z^2)+2a(a)+2a^2 = (\mathrm{OD})^2+(\mathrm{FE})^2$,

or $(\mathrm{AB})^2 + (\mathrm{AC})^2 + 4a^2 =$

or $a^2 + 4a^2 =$

or $5a^2 = (\mathrm{OD})^2+(\mathrm{FE})^2$.

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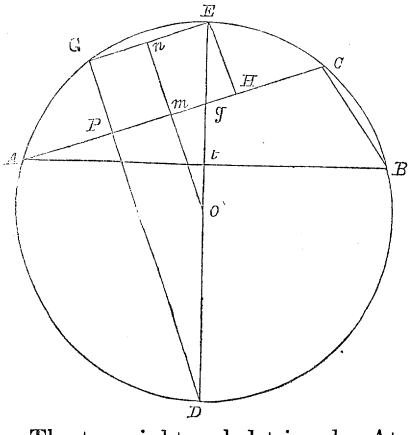
13. *If the base of a triangle be bisected by the diameter of its circumscribing circle, and from either extremity of that diameter a perpendicular be let fall upon the longer side, it will divide that side into segments, one of which will be equal to the half sum, and the other to the half difference of the sides.*



Let ABC be the triangle. Circumscribe it by a circle, and draw the diameter DE perpendicular to the base.

From the extremities D and E let fall the perpendiculars DP, EH, on the longer side AC. Produce DP to G, and complete the figure as in the margin.

The two rightangled triangles Atg, DPg are equiangular; for besides the right angle in each, they have the common angle at g; therefore the angle CAB = GDE. But equal angles at the circumference of a circle are subtended by equal chords; therefore GE = CB.

The figure GPHE is a rightangled parallelogram; for the angles at P and H are right angles by construction, and the angle DGE is a right angle, because it is in a semicircle; hence GE is parallel and equal to PH.

From the centre O, let fall the perpendicular Omn upon the lines AC and GE, and it will bisect these lines.

Now Am is the half of AC, and nE = mH is equal to the half of BC; therefore AH = $\frac{1}{2}$(AC+CB), and AP or HC is equal to half the difference of these lines.

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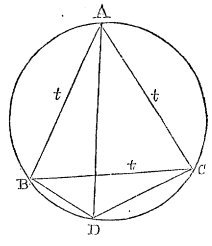
14. *A straight line drawn from the vertex of an equilateral triangle inscribed in a circle, to any point in the opposite circumference, is equal to the two lines together which are drawn from the extremities of the base to the same point.*

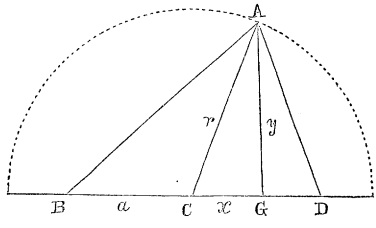


Let ABC be the equilateral triangle in the circle ABC, and designate each side by t. Take D any point in the arc BC, and join AD, BD and DC.

We are now to prove that AD=BD+DC.

Because ABDC is a quadrilateral in a circle, we have $t(AD) = t(DC)+t(BD)$; and dividing by t, gives $AD = DC+BD$. Q. E. D.

15. *If two points be taken in the diameter of a circle, equidistant from the centre, the sum of the squares of the two lines drawn from these points to any point in the circumference, will always be the same.*



Let B, D, be equally distant from the centre C. Take A any point in the circumference, and draw AG perpendicular to BD.

Put $AG = y$, $CG = x$, $AC = r$.

Now in the rightangled triangles, we have

$$r^2 = x^2+y^2, \quad \text{------ ---- ------ [1]}$$

$$(AB)^2 = y^2+(a+x)^2, \quad \text{------ ------ [2]}$$

$$(AD)^2 = y^2+(a-x)^2. \quad \text{---- ---- ---- [3]}$$

By adding [2] and [3], we have

$$(AB)^2+(AD)^2 = (2y^2+2x^2)+2a^2$$

$$= 2r^2+2a^2. \quad \text{------ ---- ---- [4]}$$

Now as r and a are both invariable, therefore the first member is invariable.

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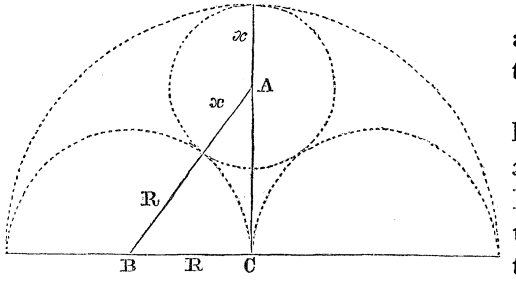
When G falls outside of D, then $(a-x)$ becomes $(x-a)$: but the square of $(x-a)$ is the same as the square of $(a-x)$; hence, in that case also, the first member of equation [4] will be invariable.

16. *If, on the diameter of a semicircle, two equal circles be described, and in the space included by the three circumferences a circle be inscribed, its diameter will be two-thirds the diameter of either of the equal circles.*



Draw the figure as represented in the margin.

Denote the unknown radius by x, and the radius BC by R; then in the right-angled triangle ACB, we have $(AC)^2+(BC)^2 = (AB)^2$.

But $AC = 2R-x$, $AB = R+x$; hence,

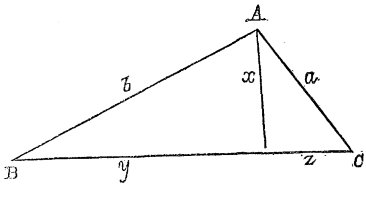
$$(2R-x)^2+R^2 = (R+x)^2,$$

or $$4R^2-4Rx+x^2+R^2 = R^2+2Rx+x^2,$$

$$4R^2 = 6Rx,$$

$$\tfrac{2}{3}R = x. \qquad \text{Q. E. D.}$$

17. *If a perpendicular be drawn from the vertical angle of any triangle to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.*



Let ABC be any triangle. Take the longer side as a base, and from the opposite angle let fall the perpendicular x upon it. Denote the sides by a and b, and the segments of the base by y and z, as shown in

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the figure. Then, by the rightangled triangles, we have

$$x^2+y^2 = b^2,$$
$$x^2+z^2 = a^2.$$

By subtraction, $y^2-z^2 = b^2-a^2$. Q. E. D.

Scholium. By factoring this equation, we have

$$(y+z)(y-z) = (b+a)(b-a),$$

or $$(y+z) : (b+a) = (b-a) : (y-z). \quad \text{---- ---- [p]}$$

This proportion is sometimes used in trigonometry, and gives the following general principle :

In any triangle, as the base is to the sum of the sides, so is the difference of the sides to the difference of the segments of the base.

From	$y+z$,
Take	$y-z$;
Difference	$2z$.

That is : *From the base, subtract the difference of the segments of the base, and divide by 2; the quotient will be the lesser segment.*

Remark. Proportion [p] is true, whatever be the relative values of y and z, and of a and b : it is therefore true when $a = b$, and $y = z$. Making this supposition, the proportion becomes

$$2y : 2b = 0 : 0,$$

or $$y : b = 0 : 0.$$

But b is always greater than y (because a hypothenuse is always greater than either side of the same triangle); therefore one *zero* may be greater than *another* zero; or, in other words, two zeros may have *any numerical ratio whatever* between them; but this is no more than saying that 0 divided by 0 may have any quotient.

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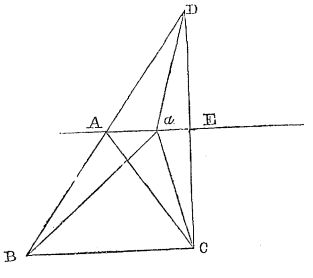
18. *The sum of the sides of any isosceles triangle is less than the sum of the sides of any other triangle of the same base and altitude.*



Let ABC be an isosceles triangle, BC the base. Through A the vertex, draw the line AE parallel with the base. Produce BA to D, making AD=AB; and join CD. CD is perpendicular to AE; for the exterior angle CAD of the triangle ABC is equal to the sum of the angles B and C; but as B and C are equal, the angle CAD is double of the angle B. But as AE is parallel to BC, the angle DAE = B; hence the angle CAE is also equal to B. Consequently the angles DAE and EAC are equal, and AD=AC, and AE is common to the two triangles DAE and AEC; therefore these two triangles are identical, and the angles AEC and AED are equal and consequently right angles. Now any point in the line AE is equally distant from the points C and D. Take any point a in the line AE, and join Ba, aC : our proposition is to prove that Ba+aC are together greater than BA+AC. The two sides or the triangle BaD are together greater than the third side BD; but BD = BA+AC; that is, Ba+aD or Ba+aC are together greater than BA+AC. Q. E. D.

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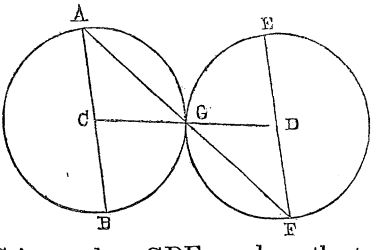
19. *If two circles touch each other externally, and parallel diameters be drawn, the straight line joining the extremities of these diameters will pass through the point of contact.*

Let the two circles whose diameters are AB and EF, touch each other at the point G, and let AB be parallel to EF. Join their centres by the line CD: this line will pass through the point of contact G.

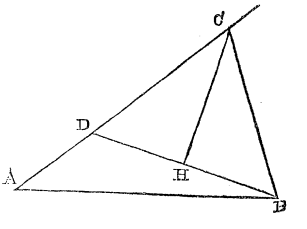


Join AG and GF. *We are now to show that* AGF *is one continued straight line.*

By reason of the parallels AB and EF, the angle ACG is equal to GDF; and as the two triangles are isosceles, therefore the angles at A, G and F, are all equal to each other. Also the angle AGC=DGF; therefore AGF is a straight line passing through the point of contact G.

20. *In any triangle, given one angle, a side adjacent to the given angle, and the difference of the other two sides, to construct the triangle.*

Let AB be the given side, A the given angle, and AD the given difference of the other two sides.



Join DB. Bisect BD in H; and from the point H, draw HC perpendicular to BD, meeting AD produced in C. Join CB, and the triangle ACB is constructed as required, for CB = CD.

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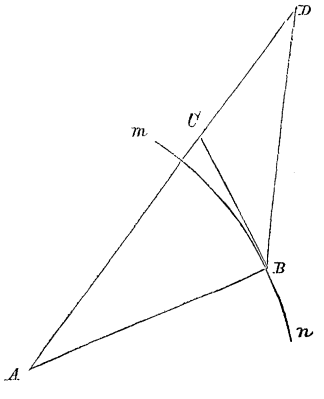
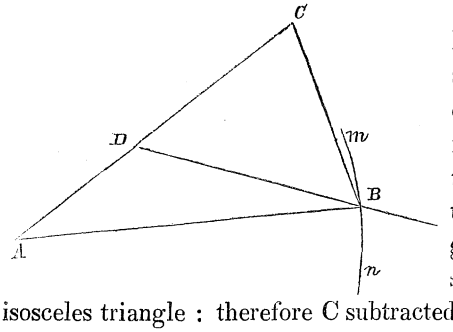
21. *In any triangle, given the base, the sum of the other two sides, and the angle opposite to the base, to construct the triangle.*



Take any point A, and draw AD equal to the given sum of the two sides. From A, with a radius equal to the given base, strike the arc mn. From D, make the angle ADB equal to half the given angle, and produce DB until it meets the arc mn in B. From B make the angle DBC equal to the angle D, and join AB. ACB is the triangle required; for CB = CD, and the angle ACB is double of the angle D, and consequently equal to the given angle.

22. *In any triangle, given the base, the angle opposite to the base, and the difference of the other two sides, to construct the triangle.*



To find a construction, let us suppose that we already have the triangle constructed, in which C is the given angle, AB the base, AC and CB the sides, and AD their given difference. Observe that CDB is an isosceles triangle: therefore C subtracted from 180°, and divided by 2, will give the angle CDB; and CDB subtracted from 180°, will give the angle ADB. We are now prepared

FOR THE CONSTRUCTION.

Draw an indefinite line AC. Take AD equal to the given difference. At D, make the angle CDB equal to $(90° - \frac{1}{2}C)$,

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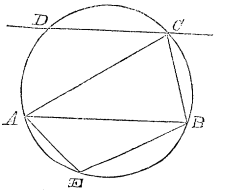
and produce DB indefinitely. From A as a centre, with a radius equal to the given base, strike an arc mn cutting DB in B. Join AB, and make the angle DBC = CDB, and ABC is constructed.

23. *In any triangle, given the base, the perpendicular, and the angle opposite to the base, to construct the triangle.*



Draw AB equal to the given base, and draw CD parallel to AB, and at a distance equal to the given perpendicular.

On the other side of AB, make the angle BAE equal to *a part* of the given angle, and the angle ABE equal to *the remaining part* of the given angle; thus forming the triangle AEB.

About the triangle AEB, describe the circle, which must either touch the line DC, or cut it in two points D and C as in the figure. Join AC, CB, or AD, DB, and we shall have the triangle required. For the angle ACB = BAE+ABE; hence ACB = the given angle.

CONSTRUCTION OF QUADRATIC EQUATIONS, WITH THEIR SOLUTION BY LOGARITHMIC SINES AND TANGENTS.

I. When the equation is in the form $x^2+px = q$.

Find an angle Q, of such a value that

$$\tan Q = \frac{2\sqrt{q}}{p}, \text{ (radius unity;)}$$

then the roots will be

$$x = +\tan \tfrac{1}{2}Q\sqrt{q}, \qquad x = -p-\tan \tfrac{1}{2}Q\sqrt{q}.$$

But this manner of presenting scientific truth is not only unsatisfactory, but perplexing. We must therefore elucidate.

Draw two lines intersecting each other at right angles. From the point of intersection, consider the upward direction *plus*, the downward direction *minus;* the righthand direction on the horizontal line *plus*, and the opposite direction *minus*.

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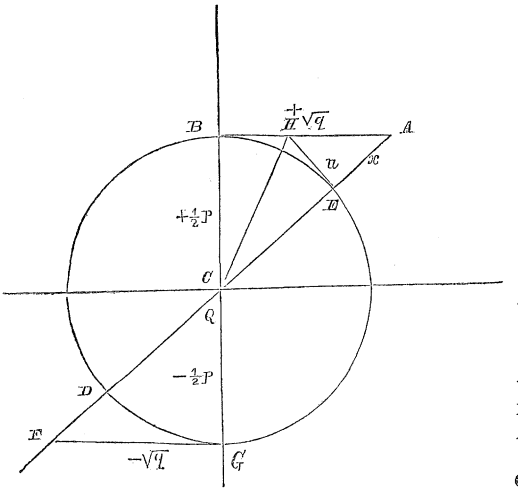
Now consider the equation $x^2+px = q$, or $(x+p)x = q$.



From C as a centre, with a radius equal to $\frac{1}{2}p$, describe a circle as represented in the figure. From B any point in the circumference of this circle, draw BA $= \sqrt{q}$. Join AC, and produce it to F (to meet the tangent GF drawn from G the opposite extremity of the diameter BG), thus forming the two equal rightangled triangles ABC and FCG. The sides of one are all *plus;* the sides of the other, all *minus.* The angle ACB, or its equal FCG, is the angle Q in our formulas. AE $= x$, *one root of the equation, and* EF *is the other.*

Because A is a point without a circle, and AD cuts the circle, and AB meets it; therefore

$$AD \times AE = (AB)^2,$$

that is, $$(x+p)x = q,$$ our equation.

In the rightangled triangle ACB, we have the two sides AB and BC, from which we can obtain the angle Q thus :

$$\tfrac{1}{2}p : \sqrt{q} = 1 : \tan Q;$$

whence $$\tan Q = \frac{2\sqrt{q}}{p}.$$

From E draw EH at right angles to AC, and join CH : then we can readily prove that the angle HCE $= \frac{1}{2}Q$.

The triangles AEH, ABC are similar, and give the proportion

$$AE : EH = AB : BC; \quad \text{----------[1]}$$

that is, $$x : EH = \sqrt{q} : \tfrac{1}{2}p. \quad \text{----------[2]}$$

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To find EH, we have the following proportion :

$$\text{CE} : \text{EH} = 1 : \tan \tfrac{1}{2}Q,$$

or $$\tfrac{1}{2}p : \text{EH} = 1 : \tan \tfrac{1}{2}Q;$$

whence $$\text{EH} = \tfrac{1}{2}p.\tan \tfrac{1}{2}Q.$$

This value of EH, put in proportion [2], gives

$$x : \tfrac{1}{2}p.\tan \tfrac{1}{2}Q = \sqrt{q} : \tfrac{1}{2}p;$$

whence $$x = \tan \tfrac{1}{2}Q\sqrt{q}.$$

As the product of the two roots must be q by the theory of equations, and as one of them is $\tan \frac{1}{2}Q\sqrt{q}$ the other must be

$$\frac{q}{\tan \frac{1}{2}Q\sqrt{q}} = \frac{\sqrt{q}}{\tan \frac{1}{2}Q} = \cot \tfrac{1}{2}Q\sqrt{q}.$$

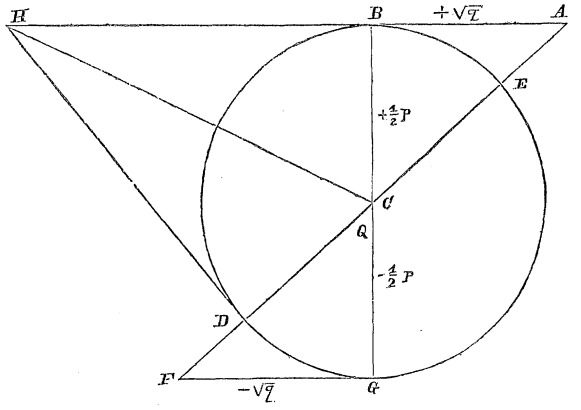
The sign to either root may be *minus;* for $\sqrt{q}$, when the root is actually extracted, may be *plus* or *minus.*

Remark. By the common method of resolving quadratics, we complete the square by squaring $(\frac{1}{2}p)$ and adding it to q, which sum is the square of AC; and if we take the root with the *plus* sign, we have AC; if with the *minus* sign, we have CF.

II. When the equation is in the form

$$x^2 - px = q, \qquad \text{or} \quad (x-p)x = q.$$

Construct the figure as here represented, and it is obvious that



$$\text{AD} \times \text{AE} = (\text{AB})^2. \text{ ----------[1]}$$

Put $x = \text{AD}$; then $(x-p) = \text{AE}$;

whence $(x-p)x = q$, the equation.

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From D, draw DH at right angles to AD, and produce AB to H. Because the two rightangled triangles ABC, ADH have the common angle A, the angle AHD = ACB = Q. Draw HC; then the angle DHC = $\frac{1}{2}$Q.

In the rightangled triangle ABC, we have, as before,

$$\tfrac{1}{2}p : \sqrt{q} = 1 : \tan Q = \frac{2\sqrt{q}}{p}.$$

From the two rightangled triangles ABC, ADH, we have

AD : DH = AB : BC; ----------[1]

that is, x : DH = $\sqrt{q}$: $\frac{1}{2}p$. ----------[2]

To find DH, we have

HD : DC = 1 : $\tan \frac{1}{2}Q$;

that is, HD : $\frac{1}{2}p$ = 1 : $\tan \frac{1}{2}Q$,

or $$\text{HD} = \frac{p}{2 \tan \frac{1}{2}Q}.$$

This value of HD, put in [2], gives

$$x : \frac{p}{2 \tan \frac{1}{2}Q} = \sqrt{q} : \tfrac{1}{2}p,$$

or $$x = \frac{\sqrt{q}}{\tan \frac{1}{2}Q} = \cot \tfrac{1}{2}Q\sqrt{q}.$$

The other root is of course (p—$\cot \frac{1}{2}Q\sqrt{q}$), or DF or AE. Also as the product of the two roots is q, and $\cot \frac{1}{2}Q\sqrt{q}$ is one root, therefore the other root must be $\frac{q}{\cot \frac{1}{2}Q\sqrt{q}} = \frac{\sqrt{q}}{\cot \frac{1}{2}Q} = \tan \frac{1}{2}Q\sqrt{q}$.

OBSERVATION.

Any quadratic equation may be geometrically represented by some rectangle; and when the sides of the rectangle are determined, the equation is solved.

The most simple representation of a quadratic is the following:

Draw any perpendicular line AO : mark one direction on it (+); the opposite direction will therefore be (−). Through any point A in this line, and at right angles to it, draw the indefinite line AE. Then if the equation is $x^2+px = q$,

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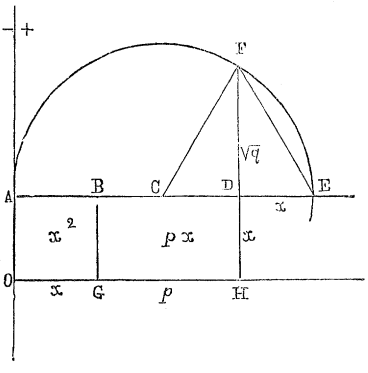
Take AB = x, BD = p; then, in a word, the first member of the equation is represented by the rectangle AH.

Take DE = DH = x; and and on AE as a diameter, describe the semicircle AFE. HD produced to F, will make DF = $\sqrt{q}$ (See Geometry). Join CF; thus forming the triangle CDF : CD = $\frac{1}{2}p$, and CF = $x+\frac{1}{2}p$.



The common method of solving quadratics consists simply in finding the hypothenuse of the rightangled triangle CDF, the sides CD and DF being given : $(CD)^2 = \frac{1}{4}p^2$, $(DF)^2 = q$; then CF = $\sqrt{q+\frac{1}{4}p^2}$, which is the second member of the equation when solved. That is,

$$x+\tfrac{1}{2}p = \pm\sqrt{q+\tfrac{1}{4}p^2}.$$

The double sign points out two triangles : CDF belongs to the *plus* sign; the other one would appear in the other semicircle, if it were drawn.

In solving the equation trigonometrically, we compute the angle FCD = Q; the angle DFE = $\frac{1}{2}Q$, and the result will be $x = \tan\frac{1}{2}Q\sqrt{q}$ as before.

The most difficult form of a quadratic is the following :

$$x^2-px = -q.$$

In respect to the signs, it is proper to remark, that in numbers and magnitudes, there is no such thing as *minus* in the sense of being less than nothing. The minus signs in this equation refer only to position; but we can give the clearest illustration by constructing the equation.

As before, draw AO (figure on next page) : one direction on it is *plus*, the other direction is *minus*. Let AB represent x, and AL represent p.

As x^2 is *plus*, we describe it on one side of the line AO; and as px is *minus*, we draw it on the other side; and their difference is q, represented by the parallelogram GLDH.

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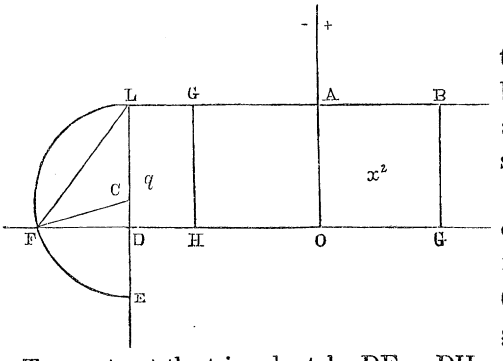

When px is less than x^2, then q will be *plus*, and will appear on the other side of AO.

To solve the equation, we must find the side LD ($=x$) of the triangle LFD.

To construct the triangle, take DE = DH, and on LE describe the semicircle; and draw LF, CF, as represented.

We have $\quad AL = p, \quad AG = x.$

Hence $\quad LG \text{ or } DH = p - x = DE;$

$$x = LD.$$

By addition, $\quad p = DE + LD = LE.$

In this case the radius of the circle is $\frac{1}{2}p$, now the hypothenuse of a triangle : in the former cases, it was a side.

When the equation is solved by the common method, the second side of the equation is in value CD of the triangle CFD, and its algebraical expression is $\sqrt{\frac{1}{4}p^2 - q}$.

To solve the equation trigonometrically, take the triangle CFD,

which gives $\quad \sin 90° : CF = \sin Q : FD;$

that is, $\quad 1 : \frac{1}{2}p = \sin Q : \sqrt{q},$

whence $\quad \sin Q = \dfrac{2\sqrt{q}}{p}.$

From the triangle LFD, we have

$$LD : DF = 1 : \tan \tfrac{1}{2}Q;$$

that is, $\quad x : \sqrt{q} = 1 : \tan \frac{1}{2}Q,$

or $\quad x = \dfrac{\sqrt{q}}{\tan \frac{1}{2}Q} = \cot \tfrac{1}{2}Q \sqrt{q}.$

The other root must be $\dfrac{q}{\cot \frac{1}{2}Q \sqrt{q}} = \dfrac{\sqrt{q}}{\cot \frac{1}{2}Q} = \tan \tfrac{1}{2}Q \sqrt{q}.$

When q has the *minus* sign, and is numerically greater than $\frac{1}{4}p^2$, the angle Q becomes *imaginary*, and consequently x becomes imaginary. *When the unknown quantity in any quadratic is imaginary, it has no real existence, and no geometrical figure can represent it.*

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Trigonometrical solutions are sometimes resorted to, when the numbers p and q are very large, or complex fractions. We give one or two

EXAMPLES.

1. From $x^2+3100x = 1800000$, to find x.

$$\text{Formulas}: \begin{cases} \tan Q = \dfrac{2\sqrt{q}}{p} = \dfrac{\sqrt{q}}{\frac{1}{2}p}; \\ x = \tan \frac{1}{2}Q\sqrt{q}. \end{cases}$$

$\frac{1}{2}p = 1550$, $q = 1800000$.

q,	log	6,255273	
$\sqrt{q}$,	log	3,127636	
$\frac{1}{2}p$,	log	3,190332	
$\tan Q$,	log	−1,937304	(radius unity.)
	Add	10,000000	(for our tables.)
$\tan Q$,	log	9,937304	.. 40° 52′ 44″ = Q;
			20 26 22 = $\frac{1}{2}Q$.
$\tan \frac{1}{2}Q$,	log	9,571333	
$\sqrt{q}$,	log	3,127636	
x,	log	2,698969	.. 500 = x. Answer.

To find the other root, take the logarithm of q already found, and from it subtract $\log x$ last found, thus :

	6,255273	
	2,698969	
log	3,556304	.. N. −3600 by changing signs.

2. From $x^2-10{,}56x = -27{,}8$, find x.

$$\text{Formulas}: \begin{cases} \sin Q = \dfrac{\sqrt{q}}{\frac{1}{2}p}; \\ x = \cot \frac{1}{2}Q\sqrt{q}. \end{cases}$$

$\frac{1}{2}p = -5{,}28$, $q = -27{,}8$.

q,	log	1,444045	
$\sqrt{q}$,	log	0,722022	
$\frac{1}{2}p$,	log	0,722634	
$\sin Q$,	log	−1,999388	(to radius unity.)
	Add	10,000000	(for our tables.)
		9,999388	.. 86° 57′ 20″ = Q;
			43 28 40 = $\frac{1}{2}Q$.

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$\cot \frac{1}{2}Q$, log 10,023085
$\sqrt{q}$, log 0,722022
x, log .... 0,745107 .. 5,56 = x.

For the other root,
1,444045
0,745107
0,698938 .. 5 = x.

Answer.

TO FIND THE ROOTS OF EQUATIONS OF THE THIRD AND FOURTH DEGREES BY CONSTRUCTION.

By the preceding pages, we perceive that equations of the second degree may be constructed by means of straight lines and circles.

Equations of the third and fourth degrees may be constructed by a circle cutting *one* of the conic sections. For simplicity, we prefer to use the parabola; and we commence by giving a general yet very brief view of the subject.

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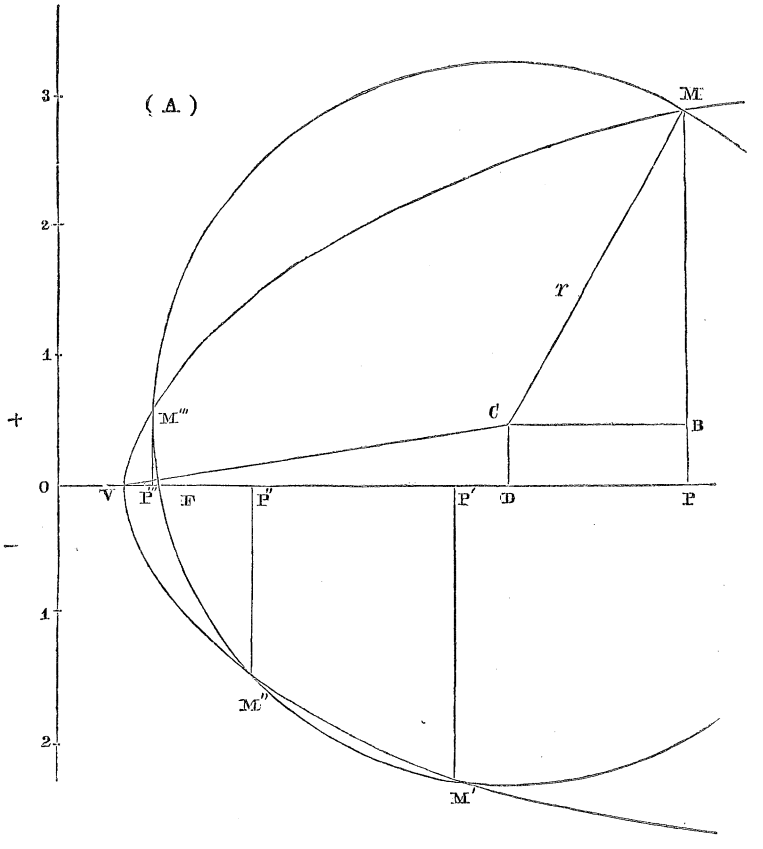
Let V be the vertex of a parabola, as represented in our figure, and F its focus. Take any point C within or without the parabola, and with any radius describe a circle. If the circle cuts the parabola, draw the ordinates from the points of intersection, as MP, M′P′, M″P″, M‴P‴; and these ordinates will be the roots *to some equation* of the fourth degree.



If the circle cuts the parabola at all, it must cut it in at least two points. If the circle does not cut the parabola, nor *touch it*, the equation to which the figure refers has no *real roots :* all its roots must be *imaginary.*

If the circle cuts the parabola in two points only, the figure

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represents some equation having *two real* and *two imaginary* roots. If the circle touches the parabola, the equation has *two equal* roots.

The reason that this figure represents some equation of the fourth degree, can be drawn from the following considerations :

The point M, being in the parabola, MP is connected with VP by the equation $y^2 = 2px$ (See Conic Sections), an equation of the *second degree*, in which MP is represented by y, VP by x, and OF by p, and p is considered a known number. The point M is also in the circle, and the equation for any point in a circle is an equation of the *second degree*; and if we combine two equations of the second degree (see Algebra, theory of equations), there will result an equation of the *fourth degree.*

Before we can form any particular equation, sufficient data must be given, that is, *a sufficient number of constant lines.*

Let VD $= a$, DC $= b$, and CM $= r$;

then DP $=$ CB $= x-a$, MB $= y-b$.

Prom the rightangled triangle MCB, we have

$$(x-a)^2+(y-b)^2 = r^2. \quad \text{---- ---- ---- ----} \quad [1]$$

If x were less than a and y less than b, then the equation will be

$$(a-x)^2+(b-y)^2 = r^2. \quad \text{---- ---- ---- ----} \quad [2]$$

But the first members of these equations do not differ in value, because $(a-x)^2 = (x-a)^2$; and either equation [1] or [2] is sufficiently general.

Expanding [1], and substituting the values of x and x^2 drawn from the equation $y^2 = 2px$, we shall have, after a little reduction,

$$y^4-(4ap-4p^2)y^2-8bpy+(a^2+b^2-r^2)4p^2 = 0. \quad \text{----} \quad [G]$$

We designate this equation [G], because it is the general and fundamental equation for all that follows on this subject.

Observe that this equation has no term containing y^3, or, what is the same thing, the coefficient to y^3 is zero; and therefore the sum of all the roots must be zero, by the general theory of equations (See Algebra). Our figure shows this : On one side of the axis VP, the roots are *positive*, and on the other side *negative;* and we see by the scale of equal parts on the line drawn through O, that MP and M‴P‴ are numerically equal to M′P′ and M″P″.

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When DC, *or* b, *represents a negative quantity, the negative roots are on the same side of the axis* VP *as the centre* C; *and the contrary.*

The parabola may be considered as given, and *constant* for any and every equation; and the circle *variable*, to correspond to any particular equation. For the sake of convenience, the value of p is taken for unity, and thus constitutes the scale of measure.*

Taking $p = 1$, immediately reduces equation [G] to

$$y^4-(4a-4)y^2-8by+(a^2+b^2-r^2)4 = 0. \quad \text{------ [L]}$$

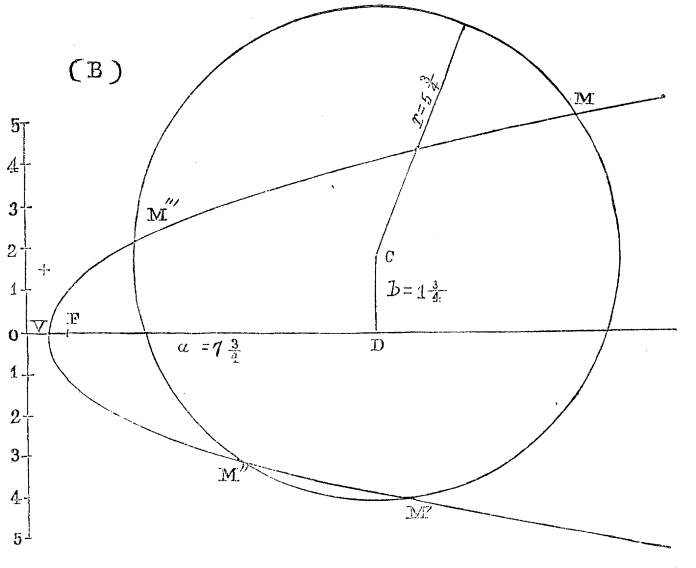
To give a practical illustration of this theory, we require the roots of the following equation *by projection :*

$$y^4-27y^2-14y+120 = 0.$$

By comparing this particular equation with the theoretical equation [L], we perceive that

$-4a+4 = -27$, $\quad -8b = -14$, $\quad a^2+b^2-r^2 = 30$;

whence $a = 7\frac{3}{4}$, $\quad b = 1\frac{3}{4}$, $\quad r = 5\frac{3}{4}$.



* It would require too much labor to describe a new parabola for every equation, as we are now compelled to do. For practice, we should have a very broad scale, on which an accurate parabola is engraved in deep-colored lines, over which tissue paper could be placed, and the constructions made with great ease and rapidity. We hope some mathematical instrument maker will take our suggestions, and make such a scale.

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Describe the parabola as represented on the preceding page. Mark the scale on the directrix *plus upward, minus downward.* Take VD $= 7\frac{3}{4}$, DC $= 1\frac{3}{4}$; and from the point C, with a radius $5\frac{3}{4}$, describe the circle which cuts the parabola at the four points M, M′, M″ and M‴, showing four real roots. If perpendiculars be let fall from these points upon the directrix, they will meet it at the respective points $+5$, -4, -3, $+2$; and thus, by this projection, we get the numerical values of the roots, and their sum is 0 as it ought to be.

In the same manner we would determine the values of a, b, and r, in any other similar equation which might be presented for solution. If the third power of the unknown quantity is contained in the equation presented, it must be taken away, by transforming the equation into another as taught in the theory of equations.

An equation with large coefficients is inconvenient for projection, as the required circle would be correspondingly large; but we can readily transform any equation into another of exactly the same form, by puttig $y = nQ$. n may be any number we choose to assign; and the roots of the equation having Q for its unknown quantity, will be the nth part of the roots of the equation containing y. For example, it will be practically impossible to determine by projection the roots of the equation

$$y^4-575y^2+2250y+25000 = 0 :$$

but put $y = nQ$; then we have

$$n^4Q^4-575n^2Q^2+2250nQ+25000 = 0,$$

or
$$Q^4-\frac{575}{n^2}Q^2+\frac{2250}{n^3}Q+\frac{25000}{n^4} = 0.$$

Now suppose $n = 5$; then

$$Q^4-23Q^2+18Q+40 = 0.$$

The roots of this equation can be found by projection, and, when multiplied by 5, will be the roots of the original equation. The roots of the transformed equation are 2, 4, -1, -5; and of the original equation, 10, 20, -5, -25.

By comparing equation [L] with figure (A), we perceive that

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the circle will pass through V if we make $a^2 + b^2 = r^2$, or $(a^2+b^2-r^2) = 0$; and if we make the circle pass through that point, the last term of equation [L] vanishes, and the equation becomes $y^4-(4a-4)y^2-8by = 0$. .... [M]

This equation, being of the fourth degree, has *four roots;* but one of the roots is obviously zero, for zero substituted in place of y will verify the equation. This also corresponds with the figure; for one of the points where the circle cuts the parabola is V, which is projected on the directrix at the point 0.

Equation [M] then reduces to

$$y^3-(4a-4)y-8b = 0. \quad \text{.... .... [N]}$$

From this equation we learn, that when a circle passes through the vertex of a parabola, the *common ordinates* to the circle and parabola will represent the roots of *some cubic equation* of the form $y^3 \pm py \pm q = 0$.

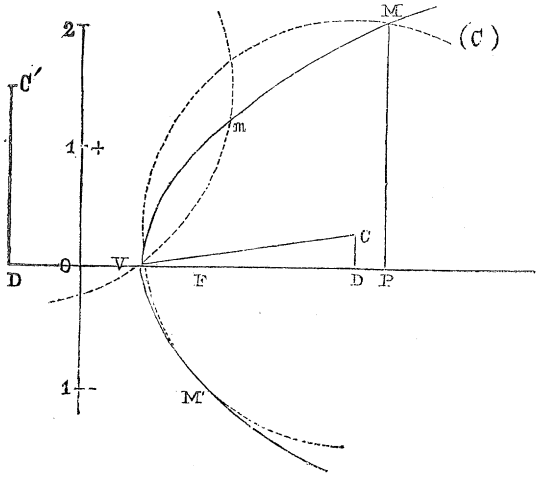
For a particular example, let us require the roots of the equation $y^3-27y-54 = 0$.

As the coefficients of this equation make too large a circle for convenience in construction, put $y = nQ$, and the equation becomes

$$Q^3-\frac{27}{n^2}Q-\frac{54}{n^3} = 0,$$

or

$$Q^3-3Q-2 = 0, \text{ by making } n = 3.$$



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By comparing this last equation with equation [N], we find

$$-4a+4 = -3, \quad \text{and} \quad -8b = -2;$$

or $$a = 1\tfrac{3}{4}, \quad \text{and} \quad b = \tfrac{1}{4}.$$

Describe a parabola as before (preceding page).

Take VD $= 1\frac{3}{4}$, DC $= \frac{1}{4}$; and from C as a centre, with CV for radius, describe a circle.

This circle cuts the parabola in M, and, as near as we can determine, touches it in M′. We perceive by the scale, or the directrix, that MP is $+2$, and M′ is -1. Hence the equation $Q^3-3Q-2 = 0$ has one positive root equal to 2, and two negative roots each equal to 1; and the roots of the original equation are thrice these numbers, or -3, -3, and $+6$.

For another example, we require the roots of the equation

$$y^3+8y-12 = 0.$$

Here $$-4a+4 = 8, \quad \text{and} \quad -8b = -12;$$

or $$a = -1, \quad \text{and} \quad b = 1\tfrac{1}{2}.$$

We may use the last parabola; and as a represents a negative quantity, it must be set off in the opposite direction from V, or *without the parabola.* Hence we take VD′ $= 1$; and from D′ we set off D′C′ $= 1\frac{1}{2}$, and C′ is the centre and C′V the radius of the circle which corresponds to this equation. The circle cuts the parabola only in one other point m, showing that the equation has only *one real* root, which is in value as near $1\frac{1}{4}$ as projection can determine. The other two roots are of course *imaginary.*

For a third example, find the roots to the equation

$$y^3+200y-27552 = 0. \quad \text{[1]}$$

On account of the large numbers, we put $y = nQ$; then

$$Q^3+\frac{200}{n^2}Q-\frac{27552}{n^3} = 0.$$

Let $n = 10$; then this equation becomes

$$Q^3+2Q-27{,}552 = 0. \quad \text{[2]}$$

Now $$-4a+4 = 2, \quad \text{and} \quad -8b = -27{,}552.$$

or $$a = +\tfrac{1}{2}, \quad \text{and} \quad b = 3{,}488.$$

Take VD $= \frac{1}{2}$, and D falls on F. Take DC $= 3{,}488$; and from C, with a radius CV, describe the circle: it cuts the para-

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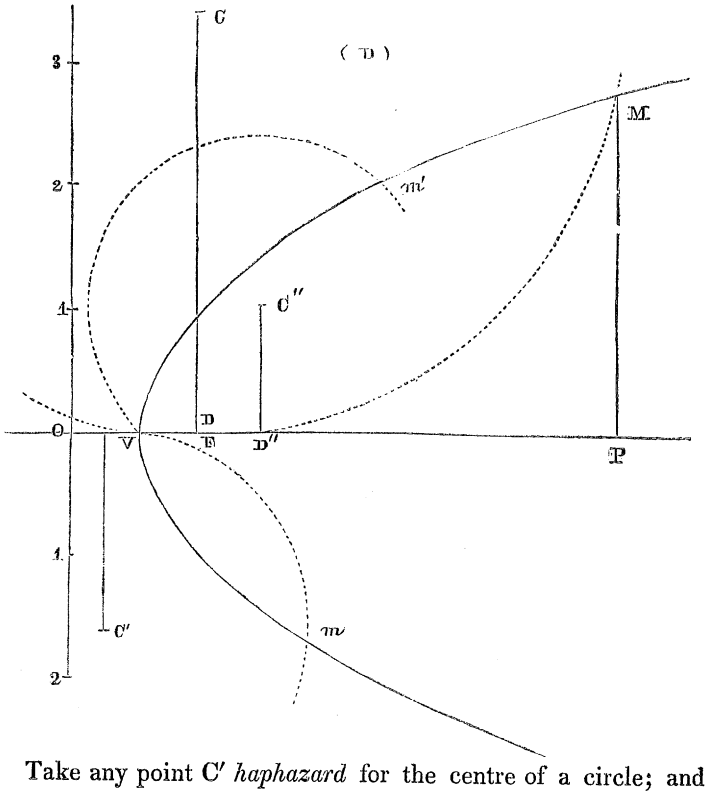
bola at M, 2,8 units from VP. Hence the root of [2] is 2,8; and of [1], 28. The other two roots are imaginary.

CONVERSELY.

We may describe any circle passing through the vertex of a parabola, and cutting it; and then find the equation which it represents.



Take any point C′ *haphazard* for the centre of a circle; and with C′V as radius, describe the arc Vm. The equation to which this circle refers is represented generally by

$$y^3 \pm py \pm q = 0;$$

in which $-4a+4 = \pm p$, and $-8b = \pm q$.

By strictly observing the *locality* of C′, we perceive that

$a = -\frac{1}{4}$, $b = -1{,}7$; whence $p = 5$, and $q = 13{,}6$.

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And the required equation is

$$y^3+5y+13{,}6 = 0;$$

an equation having one negative root equal to $1\frac{7}{10}$, as we perceive by the locality of m.

When the centre of the circle is without the parabola, the equation can have only one real root.

A circle that cuts the parabola, and does not pass through the vertex, represents an equation of the *fourth* degree, the roots of which can be found by the same principles that have just been illustrated in the last example.

BY THIS METHOD OF GEOMETRICAL PROJECTION, WE CAN FIND THE CUBE ROOT OF NUMBERS.

For example, we require the cube root of 8 *by projection.*

This requisition calls for a solution of the following equation:

$$y^3 = 8, \quad \text{or} \quad y^3+0y-8 = 0;$$

whence $-4a+4 = 0$, and $-8b = -8$,

or $a = 1$ and $b = 1$.

Take VD″ = 1 (see figure [D]), D″C″ = 1; and with C″V radius, describe the circle which again cuts the parabola in m′, which we perceive is 2 units from VP.

To show that we can determine the cube root of any number (at least approximately) *by projection*, we require the cube root of 4096; that is, find a root of the equation

$$y^3+0y-4096 = 0.$$

Put $y = nQ$; then

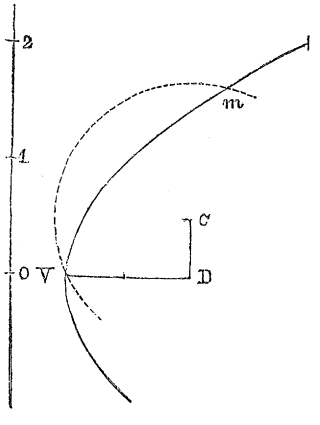
$$Q^3+0Q-\frac{4096}{n^3} = 0.$$

Take $n = 10$; then

$$Q^3+0Q-4{,}096 = 0.$$

Here $a = 1$, and $b = 0{,}512$.

Projecting the figures corresponding to these values, we find the locality of m, which gives 1,6 for the value of Q;



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and as $n = 10$, $y = 16$; or the cube root of 4096 is 16, found by this projection.

Cubic equations which can be represented by the general equation $y^3 - 3r^2y - cr^2 = 0$,

can be projected. The figure representing the projection can be found in Robinson's Geometry, page 99. r represents the radius of a circle, and c may be taken to represent any chord, which of course must be less than $2r$. If, in any particular example, c should not be small enough to fulfil this condition, transform the equation into another by putting $y = nQ$; and by taking n large enough, we may have the last term cr^2 as small as we please.

Cubic equations of this form have only one real root, and that root is a chord of one-third of the arc represented by c, and can be found by projection, *and computed trigonometrically.*

A geometrical method of determining any *magnitude*, is a sure index to its trigonometrical computation.

A surveyor who can plot his field, and properly cut it up into triangles, or place triangles about it to give it a less complex mathematical shape, can compute its contents trigonometrically: at least such is the proper supposition; yet two different surveyors might proceed differently with the same problem. So it is in respect to equations: He who can determine the roots by geometrical construction, can more accurately compute the values of the same by a trigonometrical process; but the details of the operation may be differently carried out by different persons, according to their individual knowledge, experience, and associations.

As the centre of the circle which cuts the parabola may be at any point either within or without that curve, it would require several different examples to meet the various cases that might occur; but we have neither time nor inclination to give them all: nor is it absolutely necessary, for the general principles can be drawn from one example.

As auxiliary to the trigonometrical solution of equations, we give the following table in reference to a parabola. The *unit of measure* is twice the distance from the vertex to the focus. The

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first column represents assumed distances from the vertex along the axis, called *abscissas*. The *second* column shows the corresponding *ordinates*, computed from the equation of the curve, $y^2 = 2px$; in which p is made unity, and x assumed $\frac{1}{2}$, 1, $1\frac{1}{2}$, &c. The *third* column shows the acute angles which the ordinates make with the tangents to the curve, and are computed by the proportion $2x : \sqrt{2x} = R :$ cot angle.

Abscissas.	Ordinates.	Tangent angles.
0,500	1,0000	45° 00′
1,000	1,4142	54 44
1,500	1,7320	60 00
2,000	2,0000	63 26
2,500	2,2361	65 54
3,000	2,4495	67 48
3,500	2,6457	69 17
4,000	2,8284	70 32
4,500	3,0000	71 34
5,000	3,1623	72 27
5,500	3,3166	73 13
6,000	3,4641	73 54
6,500	3,6055	74 30
7,000	3,7416	75 02
7,500	3,8730	75 17
8,000	4,0000	75 58
8,500	4,1231	76 22
9,000	4,2426	76 44
9,500	4,3589	77 05
10,000	4,4721	77 26
11,000	4,6904	77 58
12,000	4,8989	78 28
13,000	5,0990	78 54
14,000	5,2915	79 18
15,000	5,4772	79 39
16,000	5,6568	79 59

To show a method of computing by trigonometry, we adduce the following example :

$$y^3 - 6y - 5{,}6 = 0.$$

The figure on the following page represents this equation. nm is the root already determined by projection, and we see it is very

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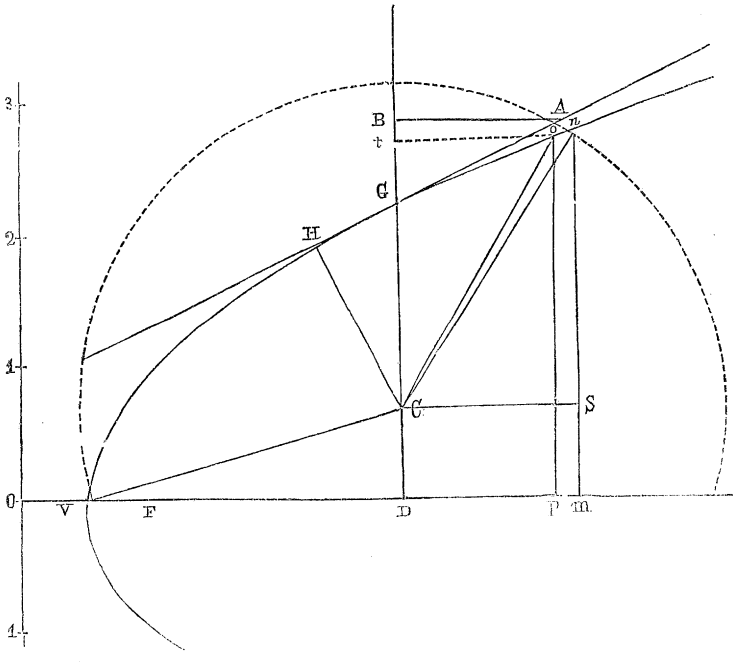
near to 3 : it is 2 *plus* some large decimal; but we cannot determine very accurately without computation.



Here $-4a+4 = -6$, and $-8b = -5{,}6$;
whence $a = 2{,}5$, and $b = 0{,}7$.

From a and b, that is, VD and DC, we find VC = 2,596, the radius of the circle.

Having VD, we compute DG from the property of the parabola. We find DG = 2,2361, and the angle CGH = 65° 54′. GC = 1,5361.

The two triangles ABG and GHC are equiangular. From GC and the angles, we compute CH = 1,4023, HG = 0,6272.

Because CH is drawn from the centre perpendicular to a chord, we can compute AH = 2,1863; whence AG = 1,5591.

Now GC : CH = AG : AB = 1,4230;
and GC : GH = AG : BG = 0,6366;
DB = DC+CG+GB = 2,8727;
VD+AB = Vp = 3,923.

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Having Vp, the equation of the parabola will give us po, which is very nearly equal to nm the root. There will be a small space between A and o; because every point in the tangent, except G, must be without the curve.

We have $\sqrt{2Vp} = po = 2{,}8013$;
whence Ao or Bt = 0,0714, Ct = 2,1013, to=BA=1,4230. Now the rightangled triangle Cto gives the angle tCo = Cop = 34° 6′ 30″, and Co = 2,5378. By the property of the parabola, we find the angle nop = 109° 39′ 45″; to which add Cop (34° 6′ 30″), and we have Con = 143° 46′ 15″. In the triangle Con, we have Co, Cn, and the angle Con; whence we can find oCn = 0° 56′ 17″.

To tCo add oCn, and we have tCn = CnS = 35° 2′ 47″.
Now the triangle CnS gives us nS = 2,1256;
To which add CD = b , = 0,7.
Therefore nm, or the root y = 2,8256.

In a similar manner we may compute the root of any other equation of the form $y^3 \pm py \pm q = 0$.*

* In the preceding equations, we have invariably represented the unknown quantity by y, because we referred to the ordinates of a parabola, and confusion might arise by using x; as x is generally supposed to represent distances along the axis, whenever a parabola is in question.

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ALGEBRA

APPLIED TO THE

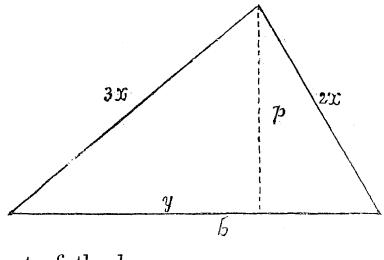
SOLUTION OF PROBLEMS IN GEOMETRY.

PROBLEM I.

To determine a triangle; having given the base, the perpendicular, and the ratio of the two sides.

From the general investigation, it is required to draw out a numerical problem, so that the base, the perpendicular, and the ratio, shall all be expressed in small rational numbers.

Let the triangle be as here represented. Take the ratio of the sides as 2 to 3; then $2x$ and $3x$ may represent the sides. Designate the base by b, and the perpendicular by p; and for immediate use, put y to represent the greater segment of the base.



Our object is to show a solution to the problem, whatever real values be assigned to b and p; and then to assign such values that the sides $2x$ and $3x$ will not be surds, or fractional numbers.

We have $y^2+p^2 = 9x^2$. ------ ------ ------ [1]

If p falls on the base, then the smaller segment is $(b-y)$; but if it falls beyond the base, then that segment is expressed by $(y-b)$. *In either case, the square of the segment is the same.* Hence,

$y^2-2by+b^2+p^2 = 4x^2$. ---- ---- ------ [2]

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Subtracting [2] from [1], gives

$$2by - b^2 = 5x^2,$$

or

$$y^2 = \left(\frac{5x^2+b^2}{2b}\right)^2.$$

This value of y^2, substituted in [1], gives

$$\frac{25x^4+10b^2x^2+b^4}{4b^2} + p^2 = 9x^2,$$

or $25x^4 - 26b^2x^2 + (b^4+4b^2p^2) = 0.$ ..... [3]

Although this equation is of the fourth degree, it is quadratic in form, and therefore easy to solve. To give it a special solution, let the quantity within the parentheses be replaced by t^2; then we have $25x^4 - 26b^2x^2 + t^2 = 0$

This expression is a binomial squared, provided

$$5x^2t = -13b^2x^2; \quad \text{(See Robinson's Alg. art. 99.)}$$

whence

$$t = \frac{-13b^2}{5}, \quad \text{or } t^2 = \frac{169b^4}{25}.$$

That is, $25x^4 - 26b^2x^2 + \frac{169b^4}{25}$, is a binomial squared; and this expression shows that the first member of equation [3] is a complete square, on condition that we make

$$\frac{169b^4}{25} = b^4 + 4b^2p^2; \quad \text{..... [4]}$$

whence

$$144b^2 = 100p^2,$$

or

$$12b = 10p. \quad \text{..... [5]}$$

Making the supposition expressed in [4], and then extracting the square root of [3], we have

$$5x^2 - \frac{13b^2}{5} = 0,$$

or

$$5x = b\sqrt{13}. \quad \text{..... [6]}$$

Now if x (and consequently $2x$ and $3x$) must be some *whole* rational number, we must put $b = 5\sqrt{13}$; then $x = 13$, $2x = 26$, and $3x = 39$. Also if $b = 5\sqrt{13}$, equation [5] gives $p = 6\sqrt{13}$.

Now we can give the problem definitely and numerically, thus:

The base of a triangle is $5\sqrt{13}$, *the perpendicular altitude* $6\sqrt{13}$, *and the less side is to the greater as* 2 *to* 3 : *Determine those sides.* Ans. 26 and 39.

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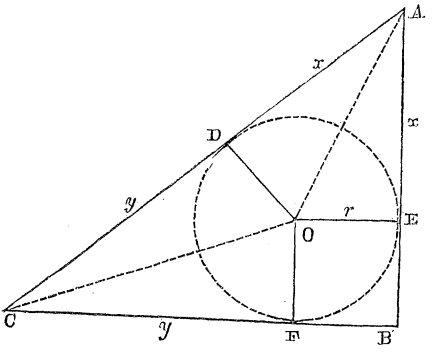
PROBLEM II.

To determine a rightangled triangle, having given the perimeter and the radius of the inscribed circle. Also give these parts in whole numbers, if possible.

Construct the figure as here represented, and we perceive two equal triangles AEO & ADO, and two other triangles CFO and CDO equal to each other. These four triangles, and the square on the radius of the circle, make up the whole triangle in question. Put AD = AE = x, CF = CD = y. Let $2a$ = the given perimeter; then $x+y+r = a$. ---- ---- ---- ---- [1]



The area of the triangle ABC may be expressed by $(x+y+r)r$, that is, by ar: it may also be expressed by $\frac{(x+r)(y+r)}{2}$; therefore $xy+(x+y+r)r = 2ar$,

or $xy = ar$. ---- ---- ---- [2]

From [1], $x^2+2xy+y^2 = a^2-2ar+r^2$,

subtract 4 times [2], $4xy = 4ar$;

$x^2-2xy+y^2 = a^2-6ar+r^2$,

or $x-y = \pm\sqrt{a^2-6ar+r^2}$.

From [1], $x+y = a-r$;

whence $x = \frac{1}{2}(a-r)\pm\frac{1}{2}\sqrt{a^2-6ar+r^2}$,

and $y = \frac{1}{2}(a-r)\mp\frac{1}{2}\sqrt{a^2-6ar+r^2}$.

Here it is obvious that x and y may be expressed in rational numbers, in case the expression under the radical sign is a complete square. It will be so, provided we make $a^2-6ar = 0$; then $a = 6r$.

This supposition gives $x = \frac{1}{2}a$, or $\frac{1}{2}a-r$;

and $y = \frac{1}{2}a-r$, or $\frac{1}{2}a$.

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Take $a = 6$, or any number of times 6, and we shall have such a triangle, that all these several parts will be whole numbers.

To find the relation which must exist between r and a, to give a triangle *the greatest possible area*, observe that equation [2] expresses the area, and the product xy will be the greatest possible when $x = y$. Making this supposition, $x = \sqrt{ar}$ from equation [2], and $x = \frac{1}{2}(a-r)$ from equation [1];

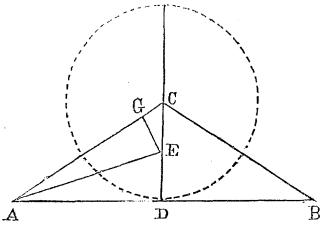
whence $$a^2-2ar+r^2 = 4ar,$$

or $$a^2+r^2 = 6ar,$$

an equation showing that a and r are incommensurable with each other in this case.

PROBLEM III.

To determine the radii of three equal circles, inscribed in a given circle, to touch each other, and also the circumference of the given circle.



Let CD = a the radius of the given circle, and make the angle ACB = 120°; then a circle inscribed in the triangle ACB will be one of the three equal circles required.

ACD = 60°, ADC = 90°; hence DAC = 30°.

Draw AE bisecting the angle A; and from E, let fall the perpendicular EG on AC; then ED = EG, and the angle GEC = 30°. ED or EG is the value of the radii required.

When the acute angles of any rightangled triangle are 30° and 60°, the side opposite 30° is half the hypothenuse.

Put ED = x; then EC = $a-x$, GE = x, and CG = $\frac{a-x}{2}$.

But $\overline{CG}^2+\overline{GE}^2 = \overline{CE}^2$, that is,

$$\left(\frac{a-x}{2}\right)^2+x^2 = (a-x)^2;$$

whence $$x = (2\sqrt{3}-3)a.$$

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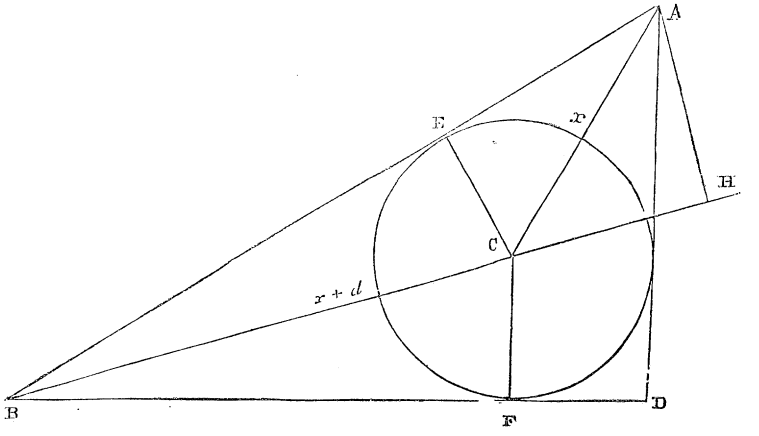
PROBLEM IV.

To determine a rightangled triangle; having given the hypothenuse, and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.

Let ABD be the triangle : AC and BC bisects the angles BAD and B.



Produce BC, and let fall the perpendicular AH. The angle ACH = 45°; then CAH = 45°, and AH = CH.

Put AH = y, AC = x, BC = $x+d$, AB = h.

In the rightangled triangle AHC, we have

$$2y^2 = x^2; \quad \text{---- ---- ------} \quad [1]$$

and in the rightangled triangle AHB,

$$(x+y+d)^2+y^2 = h^2. \quad \text{---- ---- ------} \quad [2]$$

Expanding [2], we have

$$x^2+y^2+d^2+2xy+2dx+2dy+y^2 = h^2,$$

or $$2x^2+(2x+2d)y+2dx = h^2.$$

But $y = \frac{x}{\sqrt{2}}$; whence

$$2x^2+(x+d)\sqrt{2}.x+2dx = h^2,$$

or $$(2+\sqrt{2})x^2+(2+\sqrt{2})dx = h^2,$$

or $$x^2+dx = \frac{h^2}{2+\sqrt{2}}.$$

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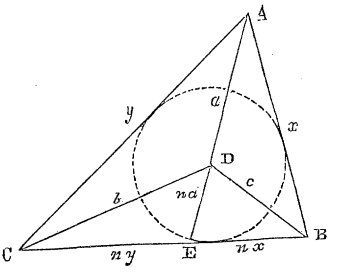
This equation shows that AC and BC can be determined, and now may be considered as known lines.

The two triangles BAH, BCE are similar, and give the proportion BA : BH = BC : BE.

The first three terms of this proportion are known : therefore BE becomes known; and by the same two triangles, CE becomes known, that is, CF and FD; and BF = BE. Hence BD is known, and the triangle ABD is fully determined.

PROBLEM V.

Given the length of three lines drawn from the centre of an inscribed circle, to the angular points of a triangle; to determine the sides of the triangle.



Let ABC be the triangle; AD, BD, CD, the three given lines. Produce AD to E. Put AD=a, and DE=na; then n is the numerical ratio between AD and DE.

The given lines bisect the angles.

When any angle of a triangle is bisected, the base is cut proportionally to the sides of the triangle. Put AB = x. Consider AE as the base of the triangle ABE; and as BD bisects the angle B, the sides are in proportion to the segments of the base : therefore nx represents the line BE. Put AC = y; then, by similar reasoning, ny = CE.

Because AE bisects the angle A of the triangle ABC, we have

$$xy = a^2(1+n)^2+n^2xy. \quad \text{(See Geom.)} \quad [1]$$

Also $$nx^2 = c^2+na^2, \quad [2]$$

and $$ny^2 = b^2+na^2. \quad [3]$$

From [1], $(1-n^2)xy = a^2(1+n)^2$,

or $$xy = \frac{a^2(1+n)}{1-n}. \quad [4]$$

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The product of [2] and [3] gives

$$n^2x^2y^2 = (c^2+na^2)(b^2+na^2). \quad \text{---- ---- ----} \quad [5]$$

Squaring [4], and multiplying the result by n^2, also gives

$$n^2x^2y^2 = \frac{a^4(1+n)^2n^2}{(1-n)^2}; \quad \text{---- ---- ---- ----} \quad [6]$$

therefore $$\frac{a^4(1+n)^2n^2}{(1-n)^2} = (c^2+na^2)(b^2+na^2). \quad \text{---- ---- ----} \quad [7]$$

This equation contains only one unknown quantity n, but it will rise to the *fourth* degree. Hence we say, that by this notation, the problem is susceptible of solution by solving an equation of the fourth degree.

By merely inspecting the figure, we perceive that in case $b=c$, AB must equal AC. This will appear by comparing equations [2] and [3]. After making this supposition, we shall have

$$nx^2 = ny^2, \text{ and } x = y.$$

PROBLEM VI.

From the three sides of any triangle, find the radius of the inscribed circle.

Take the largest side for a base, and from the opposite angle let fall a perpendicular. From the three given sides, find the perpendicular, and call it p. The base we designate by b; then the double area of the triangle will be bp. Now let x represent the required radius, and a, b and c the sides; then the double area will be expressed by $(a+b+c)x$. Hence

$$(a+b+c)x = bp,$$

or

$$x = \frac{bp}{a+b+c}.$$

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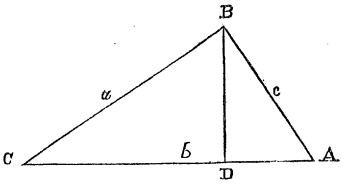
TRIGONOMETRICAL

THEOREMS AND PROBLEMS.

In all works on Surveying, we find the following rule for determining the area of a triangle when the three sides are given.

RULE. *Find the half sum of the three sides, and from said half sum take each side severally, thus obtaining three remainders. Multiply the said half sum and the three remainders together, and extract the square root of the product for the area required.*

I now propose to demonstrate this rule.



Let ABC be the triangle: its area is measured by

$$\frac{b}{2}(\text{BD}).$$

We now find the trigonometrical value of BD, by means of the triangle ADB, taking radius for unity, thus:

$$1 : c :: \sin A : \text{BD},$$

or $$\text{BD} = c \sin A;$$

whence $$\frac{b}{2}\text{BD} = \tfrac{1}{2} bc \sin A = \text{area}. \quad \text{[1]}$$

From trigonometry, $$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \text{[2]}$$

where s represents the half sum of the sides.

By a general equation of sines, we have

$$\sin A = 2 \sin \tfrac{1}{2}A . \cos \tfrac{1}{2}A;$$

whence $$\sin \tfrac{1}{2}A = \frac{\sin A}{2 \cos \tfrac{1}{2}A}. \quad \text{[3]}$$

Equating [2] and [3], and

$$\frac{\sin A}{2 \cos \tfrac{1}{2}A} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

or $$\sin A = 2 \cos \tfrac{1}{2}A . \sqrt{\frac{(s-b)(s-c)}{bc}}. \quad \text{[4]}$$

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From trigonometry, $\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$. ..... [5]

This value of $\cos\frac{1}{2}A$, put in [4], gives

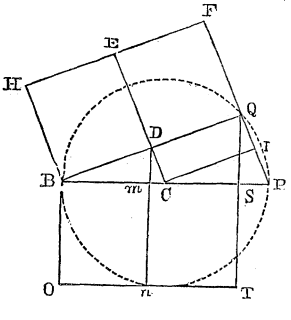
$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)},$$

or $$\tfrac{1}{2}bc\sin A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Comparing the first member of this equation with equation [1], we have, area of the triangle $= \sqrt{s(s-a)(s-b)(s-c)}$.
This expression, put in words, is the rule we were to demonstrate.

1. *Show, geometrically, that* $R(R+\cos A) = 2\cos^2\frac{A}{2}$.

Take any radius, as CP, and describe a circle. Take any arc, as PQ, and call it A. Draw PB the diameter, and join PQ and BQ. From the centre C, draw CD perpendicular to BQ, and CI perpendicular to PQ. From Q draw QST perpendicular to BS. Draw BO parallel to QT and equal to BC, and complete the parallelogram BT.



Draw Dn parallel to QT, and on BD and DQ describe squares.

$DQ = CI = \cos\frac{1}{2}A$, $CS = \cos A$, $BS = R+\cos A$.

BQP and BDC are rightangled triangles : BD = DQ; therefore Bm = mS.

Because the triangle BDC is rightangled at D, the square on BD is equal to BC.Bm, that is, equal to the rectangle Bn; but the square on BD is equal to $\cos^2\frac{1}{2}A$, that is,

$$(BD)^2 = \cos^2\tfrac{1}{2}A.$$

Double, $$2(BD)^2 = 2\cos^2\tfrac{1}{2}A;$$

that is, 2Bn or BT is equal to $2\cos^2\frac{1}{2}A$. But $BT = BS\times BC$, or $$R(R+\cos A) = 2\cos^2\tfrac{1}{2}A.$$ Q. E. D.

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2. *Demonstrate that* $\tan A+\tan B = \frac{\sin(A+B)}{\cos A.\cos B}$.

From trigonometry, we have $\tan A = \frac{\sin A}{\cos A}$,

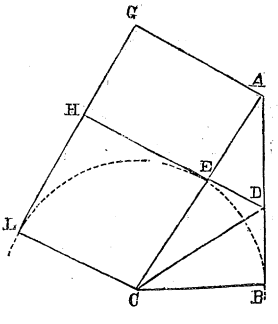
and $\tan B = \frac{\sin B}{\cos B}$.

By addition,
$$\tan A+\tan B = \frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}$$
$$= \frac{\sin A.\cos B+\sin B.\cos A}{\cos A.\cos B}$$
$$= \frac{\sin(A+B)}{\cos A.\cos B}.$$ Q. E. D.

3. *Demonstrate, geometrically, that*

$$R.\sec 2A = \tan A.\tan 2A+R^2,$$

R being the radius of the circle.



Take CB radius, and BE any arc, and call it 2A. Draw AC, secant of 2A, and AB, tan 2A.

From E, draw ED at right angles to AC; then ED = BD = tanA.

From C, draw CL at right angles to AC, & complete the parallelogram CG.

The two triangles ABC, AED are equiangular, and give the proportion

$$AE : ED = AB : BC;$$

therefore $AE.BC = ED.AB.$

Because $BC = EH = R$, therefore $AE.BC = AEHG$;

hence $AEHG = \tan A.\tan 2A.$

To each member of this equation, add ECLH, which is R^2; then the first member will be $ACLG = AC.CL = \sec 2A.R$;

whence $R.\sec 2A = \tan A.\tan 2A+R^2.$ Q. E. D.

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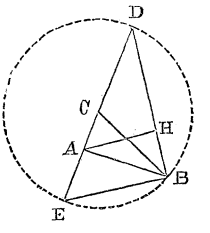
4. *Show that, in any plane triangle, the base is to the sum of the other two sides, as the sine of half the vertical angle is to the cosine of half the difference of the angles at the base.*

Let ABC be any triangle ; and, making the longer side CB radius, describe the circle : the other side AC, produced both ways, forms one diameter of the circle. Join DB and EB. DBE being in a semicircle, is a right angle.



The triangle ADB gives the proportion, AB : AD = sinD : sinABD.

Observe that AD = AC+CB; the angle D = $\frac{1}{2}$ACB; and sinABD = cosABE, because these two angles together make 90°. Hence the proportion becomes

$$AB : (AC+CB) = \sin\tfrac{1}{2}ACB : \cos ABE.$$

It remains to be shown that the angle ABE is the half difference of the angles at the base.

The angle BCD, being the exterior angle of the triangle ACB, is equal to the *sum* of the angles at the base; and as DEB is at the circumference, and stands on the same arc, DEB is the *half sum* of the angles at the base. But BAC is an exterior angle of the triangle ABE; therefore BAC = AEB+ABE. Now as AEB is the *half sum* of the angles at the base, ABE must be the *half difference*, because these two sums make the *greater* angle at the base of the triangle CAB. Hence the above proportion demonstrates the theorem.

The triangle ABE gives the following proportion :

$$AB : AE = \sin E : \sin ABE.$$

Because the angles E and D together make 90°, sinE = cosD; hence

$$AB : AE = \cos D : \sin ABE.$$

From this last proportion, we might propose the following theorem :

DEMONSTRATE *that the base of any plane triangle is to the difference of the other two sides, as the cosine of half the angle opposite the base, is to the sine of half the difference of the angles at the base.*

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Scholium. From A, draw AH parallel to EB, and consequently perpendicular to DB. Then by reason of the parallels, the angle DAH = DEB = the half sum of the angles at the base; hence HAB = ABE = the half difference of the angles at the base. If AH be made radius, DH and HB will be tangents to the angles DAH and HAB.

The triangles DEB, DAH give the following proportion :

$$DA : AE = DH : HB.$$

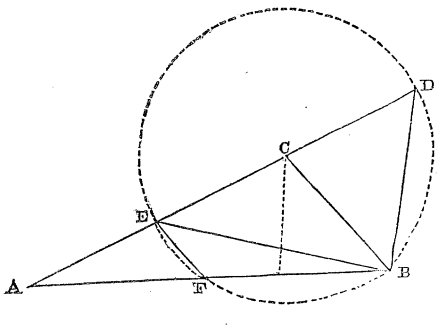
From this proportion, we may deduce the following theorem :

In any plane triangle, the sum of the sides is to their difference, as the tangent of the half sum of the angles at the base, is to the tangent of half their difference.

This theorem is demonstrated in some form in every treatise on trigonometry.

5. Give a *geometrical* demonstration of the following theorem :

The difference of two sides of a triangle, is to the difference of the segments of a third side made by a perpendicular from the opposite angle, as the sine of half the vertical angle is to the cosine of half the difference of the angles at the base.



Let ABC be the triangle. On the shorter side CB as radius, describe the circle cutting the base AB in F, and the longer side AC in E. Join EF ; produce AC to D, and join DB & BE.

AE is the difference of the sides, and AF is the difference of the segments of the base made by a perpendicular from C.

The triangle AEF gives the following proportion :

$$AE : AF = \sin AFE : \sin AEF. \quad \text{[1]}$$

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This proportion is the theorem, as will appear when we show the value of the angles.

Because EFBD is a quadrilateral inscribed in a circle,

$$\text{EFB} + \text{D} = 180°,$$

But $$\text{EFB} + \text{AFE} = 180°.$$

By subtraction, $$\text{D} - \text{AFE} = 0, \quad \text{or} \quad \text{D} = \text{AFE}.$$

In the same manner we can prove that the angle AEF = ABD; hence sinAEF = sinABD. But sinABD = cosEBF, *because the sine of an angle greater than* 90° *is equal to the cosine of the excess over* 90°; therefore proportion [1] becomes

$$\text{AE} : \text{AF} = \text{sinD} : \text{cosABE}. \quad \text{---- ---- ----} \quad [2]$$

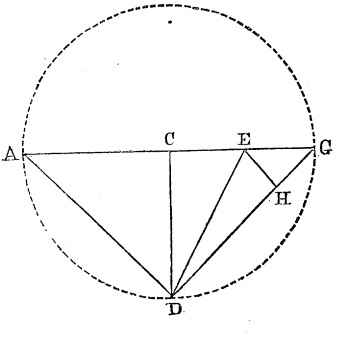
The angle D is half the vertical angle ACB, and the angle ABE is the half difference of the angles CBA and CAB, for BCD is the sum of those angles; therefore BEC = EBC = *the half sum.* Consequently EBA or ABE is *the half difference;* for half the sum of two numbers, added to half their difference, is equal to the greater number. Hence proportion [2] or [1] becomes

$$\text{AE} : \text{AF} = \sin \tfrac{1}{2}\text{ACB} : \cos \tfrac{1}{2}(\text{CBA} - \text{CAB}).$$ Q. E. D.

6. *As the greater of two sides of a triangle is to the less, so is radius to the tangent of a certain angle. Demonstrate that radius is to the tangent of the difference between this angle and half a right angle, as the tangent of half the sum of the angles at the base of the triangle, is to the tangent of half their difference.*

To obtain *that certain angle,* we must put the sides at right angles with each other.

On the longer side as radius, describe a circle. Divide the diameter into two parts AE, EG, so that AE shall be the sum of the sides; then EG will be the difference of the sides. From C, draw CD at right angles to AG. Join AD, DG, and



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draw EH parallel to AD. The angle ADG = 90°, because it is in a semicircle; hence EHG = 90°, AD = DG, EH = HG.

Let a be the greater side of any triangle, represented *in magnitude* (but not in position) by CD; and c the lesser side, represented *in magnitude* by CE : then it is plain that

$$a : c = R : \text{tanCDE}.$$

This angle, taken out of the tables, and subtracted from 45°, will give the angle EDG.

By proportional triangles, we have

$$\text{AE} : \text{EG} = \text{DH} : \text{HG or EH};$$

that is, $a+c : a-c = \text{DH} : \text{EH}.$

But $\text{DH} : \text{EH} = R : \text{tanEDG};$

whence $a+c : a-c = R : \text{tanEDG}.$

From scholium to theorem 3, we have

$$a+c : a-c = \tan \tfrac{1}{2} \text{ sum angles at base} : \tan \tfrac{1}{2} \text{ diff.};$$

whence $R : \text{tanEDG} = \tan \tfrac{1}{2}$ sum angles base : $\tan \tfrac{1}{2}$ diff.

Q. E. D.

APPLICATION.

The application of this theorem is to find the angles of triangles, when *the logarithms* of two sides, and the angle included between them, are given. It avoids the necessity of finding the sides from the logarithms; and is very useful in astronomy in finding the longitude of a planet as seen from the earth, when the position of the earth, and the position of the planet as seen from the sun, are both given.

The astronomical tables give the position of the earth from the sun, and the logarithm of its distance. They also give the position of each and every one of the planets as seen from the sun, and the logarithms of their distances. From these elements, this theorem will give the position of a planet, at any definite time, as seen from the earth. We give the following

EXAMPLE.

At noon (Greenwich time), *March* 1, 1846, *the earth, as seen from the sun, was in longitude* 160° 33′ 53″, *and the logarithm*

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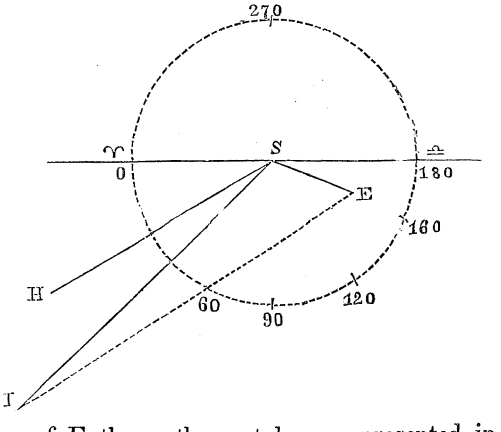
of its distance from the sun was —1,996293.* *At the same time the longitude of Jupiter, as seen from the sun, was* 46° 54′ 54″, *and the logarithm of its distance was* 0,698498. *What was the longitude of Jupiter as seen from the earth?*

(As we are not teaching astronomy, we take no account of the planet's latitude, which would affect its geocentric longitude in some slight degree.)

Let S be the position of the sun. Describe a circle around it with any radius, and take any point as ♈ for the first point, or zero point of longitude. Let SE represent the distance between the earth and sun at this particular time; and to correspond with our example, the position of E the earth must be as represented in the figure.



Place Jupiter at the point I to correspond with its given longitude and distance.

The object is to find the position of the line EI, in respect to ♈♎; or, what is the same thing, the angle ♈SH (in case SH is parallel to EI).

COMPUTATION.

From	160° 33′ 53″
Take	46 54 54
The angle ESI =	113° 38′ 59″
	180 0 0
	2) 66° 21′ 1″
½(I+SEI) =	33° 10′ 30,5″.

* The unit is the mean distance between the earth and sun, and its logarithm is 0,00000. At this time the distance was less than the mean distance, and of course the index to the logarithm is negative.

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The half difference of these angles is as yet an unknown quantity, which is our immediate object to find; and we represent it by x.

By the theorem : As 0,698498
Is to −1,996293
So is 10,000000

To tan of angle 11° 13′ 40″, whose log is 9,297795
45 0 0

33° 46′ 20″

As ------ ------ ------ ---- ------	10,000000
: tan 33° 46′ 20″ ---- ---- ------	9,825257
:: tan 33 10 30,5 ---- ---- ----	9,815417
: tan x 23 36 50 ---- ---- ----	9,640674

Diff. 9° 33′ 40″ = SIE = HSI.

Hence from lon. 46° 54′ 54″
take 9 33 40

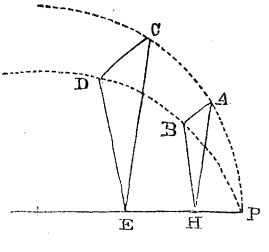
Lon. of Jupiter from the Earth, 37° 21′ 14″.

If desirable, the logarithm of EI can be readily found. A Nautical Almanac will give any number of similar examples.

6. *Two ships in the same north latitude, but* 126 *miles distant from each other, sail* 248 *miles south, and their distance from each other is then* 200 *miles. Required the latitude left, and the latitude arrived at.*

Ans. Lat. left, 83° 2′; arrived at, 78° 54′.

This problem corresponds to the supposition that the earth is a perfect sphere, and 60 miles to a degree.



(Math. Recreations.)

The ships were at A and B, and sailed to C and D.

AC = 248 miles, or 4° 8′;
AB = 126 miles, and
CD = 200 miles.

The two plane triangles ABH, CDE are similar, and give the proportion AH : CE = AB : CD.

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But AH is the sine of the arc PA, and CE is the sine of the arc PC. Let x = the arc from P to the midway point between A and C, and put a = half the arc AC; then

$$\text{AH} = \sin(x-a), \quad \text{and} \quad \text{CE} = \sin(x+a);$$

whence $\sin(x-a) : \sin(x+a) = 126 : 200,$

or $\sin x \cos a - \cos x \sin a : \sin x \cos a + \cos x \sin a = 63 : 100.$

By adding and subtracting antecedents and consequents,

$$2 \sin x \cos a : 2 \cos x \sin a = 163 : 37,$$

or $$\frac{\sin x}{\cos x} \cos a : \sin a = 163 : 37,$$

or $$\tan x : \tan a = 163 : 37;$$

whence $$\tan x = \frac{163}{37} \tan a.$$

Put $a = 2° 4'$; then,

	log tan a	= 8,557336
	log 163	= 2,212188
		10,769524
	log 37	= 1,568202
$\tan x = 9° 2'$	log	= 9,201322

PA $= (9° 2') - (2° 4') = (6° 58')$ = polar distance, or lat $83° 2'$.
PC $= (9° 2') + (2° 4') = (11° 6')$ = polar distance, or lat $78° 54'$.

7. *A ship at sea came in sight of two islands, one bearing north, the other west. After sailing* N 25° W 8 *miles, the ship was at equidistance from the islands; and after sailing* 3 *miles further on the same course, she was in a line between the islands. What was her distance from the islands when they were first seen?**

* We adduce this example to show the power of algebraical analysis as applied to geometry : a solution purely geometrical, we could never attain.

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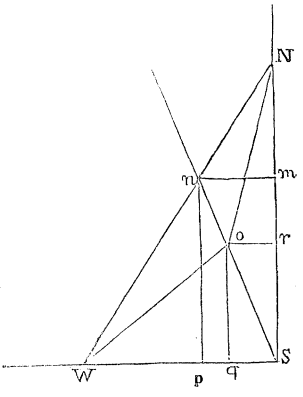

Let S be the position of the ship, N the northern island, W the western one. Sn is the direction the ship sails : So = 8 miles; on = 3 miles. Put SN=y, SW=x, So=a, Sn=b, and the angle nSm = 25° = A; then

$$\begin{aligned} \text{Sn} &= a\cos\text{A}, \\ \text{or} = \text{Sq} &= a\sin\text{A}; \\ \text{Sm} = \text{np} &= b\cos\text{A}, \\ \text{nm} = \text{Sp} &= b\sin\text{A}; \\ \text{Nr} &= y - a\cos\text{A}, \\ \text{Wq} &= x - a\sin\text{A}. \end{aligned}$$

Because o is at equal distances from N and W, we have

$$(x-a\sin\text{A})^2+a^2\cos^2\text{A} = (y-a\cos\text{A})^2+a^2\sin^2\text{A}. \quad [1]$$

By reduction, we find

$$x^2-2ax\sin\text{A} = y^2-2ay\cos\text{A}. \quad [2]$$

By similar triangles, we have

$$\text{Wp} : \text{np} = \text{WS} : \text{SN};$$

that is,

$$x-b\sin\text{A} : b\cos\text{A} = x : y; \quad [3]$$

whence

$$y = \frac{b\cos\text{A}.x}{x-b\sin\text{A}}.$$

This value of y, put in equation [2], gives

$$x^2-2ax\sin\text{A} = \frac{b^2\cos^2\text{A}.x^2}{(x-b\sin\text{A})^2} - \frac{2ab\cos^2\text{A}.x}{x-b\sin\text{A}},$$

or

$$(x-2a\sin\text{A})(x-b\sin\text{A}) = \frac{b^2\cos^2\text{A}.x}{x-b\sin\text{A}} - 2ab\cos^2\text{A}.$$

Here we have an equation of the third degree, and its solution will give SW; but as we care nothing for the numerical result, we here leave it.

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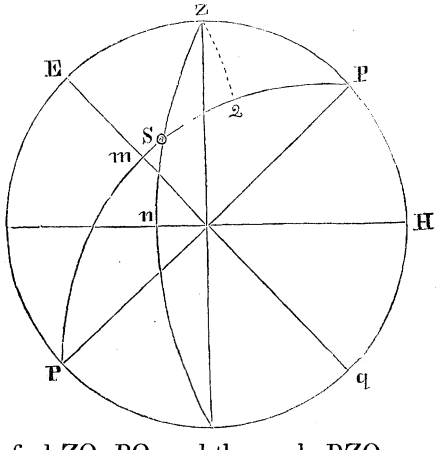
ASTRONOMICAL PROBLEMS.

1. *In latitude* 40° 48′ *north, the sun bore south* 79° 16′ *west at* 3h 38m P. M. *apparent time. Required his altitude and declination at that time.*

In this example there is nothing difficult; and any one who can construct the figure, can solve the problem.

The time from noon gives the angle ZPS = 54° 30′ : EZ = 40° 48′; hence ZP = 49° 12′.

The object of the problem is to find Sm ♄ and Sn.



In the rightangled spherical triangle PQZ, we have the side PZ and the angles P and Q; and therefore we can find ZQ, PQ, and the angle PZQ.

Because we have the bearing of the sun, we have the angle PZS = 100° 44′; from which subtract PZQ, and we shall have QZS; and having the side ZQ and the right angle Q, we can obtain all the parts of the rightangled triangle QZS. Then the complement of ZS is the sun's altitude, and PQ+QS is the sun's polar distance, which is the complement of the declination.

In this case, altitude = 36° 46′; sun's declination, 15° 32′ N.

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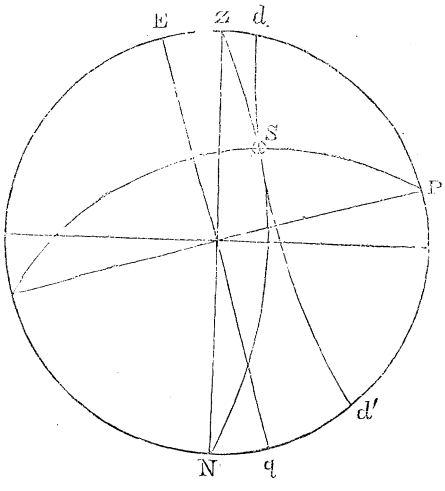
2. *In latitude* 16° 4′ *north, when the sun's declination was* 23° 4′ *north; required the time in the afternoon, and the sun's altitude and bearing, when his azimuth neither increases nor decreases.*

Ans. Apparent time 3h 9m 26s P. M.; altitude, 45° 1′; bearing north 73° 16′ west.



Let dd′ be the small circle of the sun's declination.

The problem requires the point where the great circle ZSN will touch the small circle dd′. Let it touch at S; and then we have the right-angled spherical triangle PSZ, in which PS = 66° 58′, and PZ = 73° 56′.

Spherical trigonometry gives the proportion

$$R : \cos PS = \cos ZS : \cos PZ,$$

or $$R : \sin(23° 2′) = \cos ZS : \sin(16° 4′);$$

whence ZS = 44° 59′, or the altitude = 45° 1′;

PZS = 73° 16′, and ZPS = 47° 21′ 30″ = 3h 9m 26.

3. *In latitude* 16° 40′ *north, when the sun's declination was* 23° 18′ *north, I observed it twice in the forenoon bearing north* 68° 30′ *east. Required the times of observation, and the sun's altitude at each time.*

Ans. Times, 6h 15m 40s, and 10h 32m 48s A. M.; altitudes, 9° 59′ 36″, and 68° 29′ 42″.

The last figure will serve for the problem, with a little aid of imagination. In this example the circle ZSN must cut the small circle dd′ in two points, and the angle PZS is given = 68° 30′. No further remarks are necessary.

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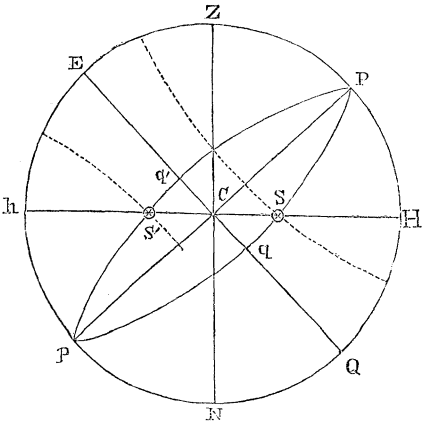
4. *Given the latitude, the sun's declination, or the declination of any planet or star, to find the semidiurnal arc for the sun, planet or star.*

Let Hh be the horizon, EQ the equator, Pp the six o'clock meridian, S or S′ the position of the sun or star, according as its declination is north or south.



The portions of the equator represented by Cq, Cq′, changed into time, and added to six hours if the latitude and declination are both north, or subtracted from six hours if one is north and the other south; the sum or difference, as the case may be, will be the *interval* between the meridian position of the sun, planet or star, and its rise from or fall to the horizon.

We find the arcs Cq, Cq′ by means of the triangles CqS, Cq′S′. Sq is the declination, and the angle SCq is the complement of the latitude.

By trigonometry, $R.\sin Cq = \tan Sq.\cot SCq.$

When radius is unity, this equation becomes

$$\sin Cq = \tan D.\tan L;$$

where D represents the declination whatever it is, and L the latitude.

We give the preceding as a *lemma* to the following problem :

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4. *In* 1850, *the right ascension of Spica was* 13h 17m 18s, *its south declination* 10° 22′ 37″; *the right ascension of Arcturus was* 14h 8m 49s, *and north declination* 19° 57′ 56″. *A traveller to California observed that Spica set* 2h 26m 14s *before Arcturus* (*allowance made for refraction*). *What was his north latitude?*

Let L = the latitude; d = 10° 22′ 37″, D = 19° 57′ 56″.

The star Spica came upon the observer's meridian at some definite time that we denote by M.

Then $$M + \left(6 - \frac{\mathrm{Cq}'}{15}\right) = \text{the time Spica set;}$$

$$M + 0\text{h } 51\text{m } 31\text{s} + \left(6 + \frac{\mathrm{Cq}}{15}\right) = \text{the time Arcturus set.}$$

Observe that 0h 51m 31s is the difference of the right ascensions of the two stars. Cq and Cq′ are right ascension arcs, and an arc is changed into time by dividing it by 15; of course, making *one* hour the unit.

By subtracting the time Spica set, from the time Arcturus set, we shall obtain an expression equal to 2h 26m 14s;

that is, $$0\text{h } 51\text{m } 31\text{s} + \frac{\mathrm{Cq}}{15} + \frac{\mathrm{Cq}'}{15} = 2\text{h } 26\text{m } 14\text{s}, \quad \text{---- ---- [1]}$$

or $$\mathrm{Cq} + \mathrm{Cq}' = 15(1\text{h } 34\text{m } 43\text{s}). \quad \text{---- [2]}$$

Equation [1] expresses *time* : equation [2] expresses *arc*. When we divide arc by 15, we obtain time : when we multiply time by 15, we obtain arc; hence 1 hour multiplied by 15, gives 15°. Therefore the sum of the arcs

$$\mathrm{Cq} + \mathrm{Cq}' = 23° \; 41'. \quad \text{---- ---- ------ [3]}$$

By the preceding problem,

$$\sin \mathrm{Cq}' = \tan d . \tan L, \quad \text{---- ---- ---- [4]}$$

and $$\sin \mathrm{Cq} = \tan D . \tan L. \quad \text{---- ------ [5]}$$

Put a = 23° 41′, and observe that

$$\mathrm{Cq}' = a - \mathrm{Cq};$$

whence $$\left.\begin{aligned} \sin \mathrm{Cq}' &= \sin(a - \mathrm{Cq}) \quad \text{---- ----} \\ &= \sin a \cos \mathrm{Cq} - \cos a \sin \mathrm{Cq}. \end{aligned}\right\} \quad [6]$$

From [5], $$1 - \sin^2 \mathrm{Cq} = 1 - \tan^2 D . \tan^2 L;$$

that is, $$\cos^2 \mathrm{Cq} = 1 - \tan^2 D . \tan^2 L.$$

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The values of sinCq′, sinCq, and cosCq$=\sqrt{1-\tan^2 D.\tan^2 L}$, put in [6], gives

$$\tan d.\tan L = \sin a\sqrt{1-\tan^2 D.\tan^2 L} - \cos a.\tan D.\tan L. \quad [7]$$

By transposition and division,

$$\left(\frac{\tan d+\cos a.\tan D}{\sin a}\right)\tan L = \sqrt{1-\tan^2 D.\tan^2 L}.$$

Let this coefficient be represented by m; then

$$m^2\tan^2 L = 1-\tan^2 D.\tan^2 L.$$

By division, $$m^2 = \frac{1}{\tan^2 L}-\tan^2 D,$$

or $$m^2+\tan^2 D = \cot^2 L, \quad \text{because } \frac{1}{\tan^2 L} = \cot^2 L.$$

We are now left to apply this equation.

First find m, which is $\left(\frac{\tan d}{\sin a}+\frac{\tan D}{\tan a}\right)$. We take the logarithms from the table of logarithmic sines and tangents, with 10 dropped from the index, because our equation refers to radius unity, thus:

$\tan d$	$-1{,}262978$	$\tan D$	$-1{,}560270$
$\sin a$	$-1{,}603882$	$\tan a$	$-1{,}642090$
0,3623	$-1{,}559096$	0,8283	$-1{,}918180$

Whence $m = 0{,}3623+0{,}8283 = 1{,}1906;$

$m^2 = 1{,}4175;$ $\tan^2 D = 0{,}1320.$

Therefore $\cot^2 L = 1{,}5495;$

$\cot L = 1{,}2447;$ $\log +10 = 10{,}095000,$

corresponding to latitude 38° 46′ north.

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6. *On the* 14*th of November* 1829, *the star Menkar* (*whose right ascension was* 2h 53m 21s, *and declination* 3° 24′ 52″ *north*) *rose* 48m 3s *before Aldebaran* (*whose right ascension was* 4h 26m 7s, *and declination* 16° 19′ 31″ *north*) : *From what latitude was the observation made?* Ans. Lat. 39° 33′ north.

Let M be the time Menkar was on the observer's meridian;

then $$M-\left(6+\frac{\text{arc}}{15}\right) = \text{time Menkar rose},$$

and $$M+r-\left(6+\frac{\text{Arc}}{15}\right) = \text{time Aldebaran rose :}$$

Diff. $$r-\frac{\text{Arc}}{15}+\frac{\text{arc}}{15} = 48\text{m } 3\text{s}.$$

The difference of the right ascensions of the two stars is represented by r. Hence r = 1h 32m 46s, and

$$\frac{\text{Arc}}{15}-\frac{\text{arc}}{15} = 0\text{h } 44\text{m } 43\text{s} = 0{,}74527;$$

whence $$\text{Arc}-\text{arc} = (0{,}74527)15 = 11^\circ\ 10'\ 45''.$$

For more distinctness, put $a = 11^\circ\ 10'\ 45''$, x=Arc, y=arc;

then $$x = a+y,$$

and $$\sin x = \sin(a+y)$$
$$= \sin a\cos y+\cos a\sin y.$$

Let $d = 3^\circ\ 24'\ 52''$, $D = 16^\circ\ 19'\ 31''$, and L = the required latitude; then $\sin x = \tan D.\tan L$, and $\sin y = \tan d.\tan L$.

By proceeding as in the previous problem, we find

$$\cot^2 L = m^2+\tan^2 d,$$

in which $$m = \frac{\tan D}{\sin a}-\cot a.\tan d.$$

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DIVISION OF LANDS.

The two following problems are from GUMMERE's *Surveying*, and are considered very difficult.

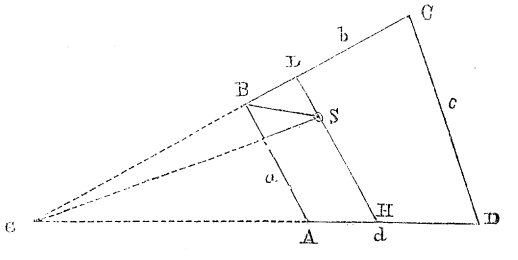
I. *There is a piece of land bounded as follows :*

Beginning at the southwest corner; thence,

1.	N 14° 00′ W,	Distance	15,20	chains	$= a$;
2.	N 70 30 E,	"	20,43	"	$= b$;
3.	S 6 00 E,	"	22,79	"	$= c$;
4.	N 86 30 W,	"	18,00	"	$= d$.

Within this lot there is a spring; the course to it from the second corner is S 75° E, *distance* 7,90 *chains. It is required to cut off ten acres from the west side of this lot, by a line running through the spring. Where will this line meet the fourth side, that is, how far from the first corner?* Ans. 4,6357 chains.

First make a draft of the field. It is as here represented.



Produce the sides b and d, the second and fourth, until they meet at G. Let S be the position of the spring, and join SG. We may or may not find the contents of the field* : it is not necessary for the location of the line LSH.

* When we have *four sides only*, and all the angles, as in this field, the best method of finding the contents is by conceiving it to be two triangles. Thus, in this case, the area is represented by

$$\tfrac{1}{2}\,ab.\sin \mathrm{ABC} + \tfrac{1}{2}\,dc.\sin \mathrm{CDA}.$$

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It is necessary to find the area of the triangle ABG. Conceive a meridian line run through B; then we perceive that the angle ABG = 14° + 70° 30′ = 84° 30′. Conceive also a meridian line to be run through A, and then we perceive that the angle BAG = 86° 30′ − 14° = 72° 30′; whence AGB = 23°. With the angles and the side AB=15,20, we readily find AG=38,72, BG = 37,10, and the area AGB = 280,65 square chains.

It is necessary to find the line GS and the angle BGS. From the given direction of the lines BC and BS, we find the angle GBS = 145° 30′; and then from the triangle GBS, we find BGS = 5° 51′ 30″, and GS = 43,83. Also we have the angle SGA = 17° 8′ 30″.

To the area of the triangle AGB = 280,65, add 10 acres, or 100 square chains: then the area of the triangle GLH must equal 380,65 square chains; but GL and GH are both unknown. Put GL = y, GH = x: then we shall have the equation

$$xy \sin 23° = 2(380,65). \quad \text{[1]}$$

It is obvious that the sum of the two triangles LGS, SGH is equal to the triangle GLH.

Put GS = m = 43,83, $\sin 23° = P$, $\sin(17° 8′ 30″) = Q$, $\sin(5° 51′ 30″) = R$, and 2(380,65) or 761,3 = a: then we have

$$Pxy = a, \quad \text{[1]}$$

and

$$Rmy + Qmx = a. \quad \text{[2]}$$

From [1], $y = \frac{a}{Px}$. This value put in [2], gives

$$\frac{aRm}{Px} + Qmx = a; \quad \text{[3]}$$

whence

$$x^2 - \frac{a}{mQ}x = -\frac{aR}{PQ}. \quad \text{[4]}$$

We now find the numerical values of $\frac{a}{mQ}$ and $\frac{aR}{PQ}$ by logarithms, as follows:

As our radius is unity, we diminish the indices of the logarithmic sines by 10.

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	$\log a$, 2,881556	$\log a$,	2,881556
$\log m$,	1,641771	$\log R$,	−1,008880
$\log Q$,	−1,469437		1,890436
	1,111208 .. 1,111208		
	58,932 .. 1,770348	$\log P$,	−1,591878
		$\log Q$,	−1,469437
			−1,061315 .. −1,061315
	674,72		2,829121

Equation [4] now becomes

$$x^2 - 58{,}932x = -674{,}7;$$

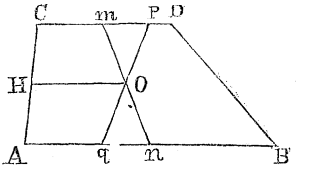
whence x = GH = 43,366 : from which subtract GA = 38,72, and we have AH the distance required = 4,646, which differs from the given answer about *one link* of the chain.

LEMMA.

Find the point in any trapezoid, through which any straight line which meets the parallel sides will divide the trapezoid into two equal parts.

Let ABCD be the trapezoid.

Bisect the parallel sides AB and CD in the points n and m. Join mn, and bisect mn in O, and O is the point required.



Any line meeting the parallel sides, and passing through O, will divide the trapezoid into two equal trapezoids. It is obvious that the line mn divides the figure into two equal parts, because the sum of the parallel sides is the same in each. Now draw any other line through O, as pOq : the trapezoid pqBD = mnBD; because the triangle Omp = qOn, and one triangle is cut off and the other is put on at the same time. The triangles are equal, because mO = On, the angle pmO = Onq, and the opposite angles at O are equal : therefore mp = nq; and whatever *more*

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than Cm is taken on one side, the equal quantity qn *less* than An is taken on the other side.

Another method of finding the point O, is to bisect AC in H, and draw HO parallel to AB or CD, and equal to one-fourth the sum of AB and CD.

II. PROBLEM.

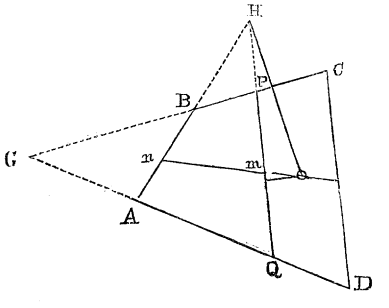
There is a piece of land bounded as follows :

Beginning at the westernmost point of the field; thence,

1. N 35° 15′ E, 23,00 chains;
2. N 75 30 E, 30,50 " ;
3. S 3 15 E, 46,49 " ;
4. N 66 15 W, 49,64 " .

It is required to divide this field into four equal parts, *by two lines, one running parallel to the third side, the other cutting the* first *and* third *sides. Find the distance of the parallel line from the first corner measured on the fourth side, and the bearing of the other line.*

Gummere's answer : Distance to the parallel, 32,50 chains;
Bearing of the other side, S 88° 22′ E.



Construct the figure as here represented, according to the directions and distances; AB being the first side, &c.

Produce the sides AD, BC to meet in G. The angle GAB = 66° 15′ + 35° 15′ = 101° 30′, and GBA = 75° 30′ − 35° 15′ = 40° 15′; hence the angle G = 38° 15′. With the side AB and the angles, the sides and area of the triangle GAB can be readily found.

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The area of the field is equal to

$$(11{,}5)(30{,}5)\sin(139^\circ\ 45') + (24{,}82)(46{,}49)\sin 63^\circ.$$

This expression gives 1254,86 square chains for the whole area, and of course one of the four equal divisions is 313,715 square chains.

The area of the triangle GAB = 270,5 square chains; GA = 24,05, GB = 36,40 : hence GD = 73,69, GC=66,90.

Let PQ be drawn parallel to DC, and let it divide the field into two equal parts; then the two triangles GDC and GQP are equiangular.

The area of the triangle GDC = 270,5+1254,86 = 1525,36;
the area of the triangle GQP = 270,5+ 627,43 = 897,93.

Put GQ = x; then

$$(73{,}69)^2 : x^2 = 1525{,}36 : 897{,}93 : \qquad [x=56{,}54.]$$

whence GQ−GA = AQ = 56,54−24,05 = 32,49.

We have now all the parts of the trapezoid QDCP, and can find the point O within it, through which any line will divide it into two equal parts (See lemma). But a line must be drawn from O in such a direction as to divide the area ABPQ into two equal parts. Let On be that line. Join OH. OH and the angle OHm can be computed; also BH and the angle BHm, and the area of the triangle BHP; to which add 313,71, and we shall have the whole area Hnm.

Put Hn = x, Hm = y, HO = a, area Hnm = A;

then $\frac{1}{2}xy\sin\mathrm{BHP} = A$, ---- ---- ---- ---- [1]

and $\frac{1}{2}ax\sin\mathrm{BHO} - \frac{1}{2}ay\sin\mathrm{OHm} = A$. ---- ---- ---- ---- [2]

From [1] and [2] we can compute x and y, which give the points n and m, and therefore the direction nm.

The angle AHQ or BHP is found thus :

GAB = 101° 30′; therefore HAQ = 78° 30′. Because PQ is parallel to CD, the angle AQH = D = 63°; whence the third angle of the triangle AQH, that is, AHQ = 38° 30′.

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Having the angles and the side AQ = 32,49 of this triangle, we find QH = 51,16 and AH = 46,30. But AB = 23; therefore BH = 23,30.

By the aid of the triangle GQP, we find PQ = 35,89; hence QH—QP = HP = 15,27.

Now the area of the triangle BHP is expressed thus :

$$\tfrac{1}{2}(23)(15{,}27)\sin(38^\circ\ 30') = (11{,}5)(15{,}27)\sin(38^\circ\ 30')$$
$$= 109{,}32 \text{ sq. chains.}$$

To this add BnmP = 313,71 " ;

Area Hnm, designated by A = 423,03 sq. chains.

By means of the two triangles HBP and Hnm, we have the following proportion, which is in fact but a modification, or rather a simplification of equation [1] :

$$xy : (23{,}3)(15{,}27) = 423{,}03 : 109{,}32;$$

whence $xy = 1390{,}6.$ ---- ---- ------ [3]

With the value of A, equation [2] becomes

$$x \sin\text{BHO} - y \sin\text{OHm} = \frac{846{,}06}{a}. \quad \text{---- ---- ------ [4]}$$

We must now find the value of the angles BHO and OHm, and the numerical value of HO, designated by a.

The area of the trapezoid PQDC is 627,42 square chains, and its parallel sides are DC = 46,49 and PQ = 35,89. Half their sum is 41,19 : therefore divide 627,42 by 41,19, and we have 15,22 for the perpendicular distance between DC and PQ; and from O to PQ is 7,61.

The distance from O to the middle point of QD is one half of 41,19, or 20,59; but this is greater by 3,89 than the distance from Q to the perpendicular from O; that is, from Q to the perpendicular is 20,59 less 3,89, or 16,70.

From OH = 51,16, take 16,70, and we have 34,49 for the longer side of a rightangled triangle whose hypothenuse is HO.

Now we have 34,46 : 7,61 = rad : tanOHm.

This proportion gives OHm = 12° 27′, and HO = 35,29 = a.

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To BHP = 38° 30′, add 12° 27′, and we have BHO = 50° 57′.

Now equation [4] becomes

$$\sin(50^\circ\ 57')x - y\sin(12^\circ\ 27') = \frac{846{,}06}{35{,}29}.$$

Substituting the value of y from equation [3], we have

$$\sin(50^\circ\ 57')x - \frac{1390{,}6.\sin(12^\circ\ 27')}{x} = 23{,}97;$$

$$x^2 - 23{,}97x = \frac{1390{,}6.\sin(12^\circ\ 27')}{\sin(50^\circ\ 57')}.$$

log 1390,6	3,143020
sin(12° 27′)	−1,333624
	2,476644
sin(50° 57′)	−1,884254
log 391,2	2,592390

$$x^2 - 23{,}97x = 391{,}2; \qquad x = 35{,}1.$$

From HA = 46,30, take 35,1, and we have An = 11,29. The triangle HnO is nearly isosceles : HO = 35,29; Hn = 35,1. I make the angle HnO = 64° 57′, which gives the direction of nO south 79° 48′ east.

I have made an error in numerical computation, or this part of Gummere's answer is erroneous.

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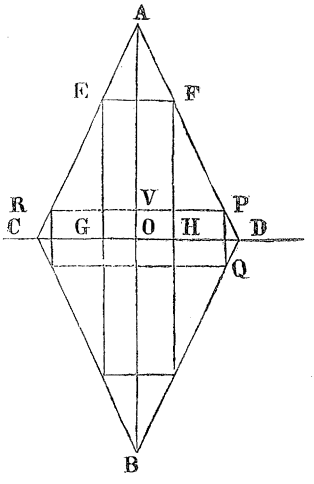
I received the two following problems from Mr. D. M. KIMBALL, of Warren, Massachusetts.

PROBLEM I.

Suppose five men own a grindstone in equal shares. The stone is 7 feet 9 inches in diameter, 4 inches thick at the eye, and the eye-hole four inches square, and tapers to a point at the outer edge. Now if each man were to grind off his proportional share separately, how many inches will each one diminish its diameter?

In the first place we must compute the solidity of the stone, and divide it by 5.



Let AB represent the diameter of the stone, and CD its axis; then ACBD will represent a diametral section.

If the plane ACBD revolve on its axis CD, it will generate two equal cones; and if from these we take the cylinder QR and the two small cones on its ends, we shall have the solidity of the stone.

AB = 93 inches; AV = 44,5; VP = 2, and we find OD by the following proportion :

$$44{,}5 : 2 :: 46{,}5 : \text{OD} = \frac{93}{44{,}5}.$$

The expression for the solidity of one cone is $0{,}7854(93)^2\frac{31}{44{,}5}$;

for two cones, $\frac{0{,}7854(93)^2 62}{44{,}5} = 9464{,}3$ cubic inches.

Although the eye-hole is *square*, there is a cylindrical waste space about the axis, whose diameter is the *diagonal* of that square. The square of this diameter is 32; hence the cylinder RQ is expressed by 0,7854(32)4. To this add the two small cones at its ends. The altitudes of these cones are

$$\text{OD}-\text{PV} = \frac{93}{44{,}5} - \frac{89}{44{,}5} = \frac{4}{44{,}5}.$$

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For two cones, the expression is $\frac{0{,}7854(32)8}{133{,}5} = 1{,}506.$

The cylinder RQ, plus the cones = 102,037 cubic inches.

From the whole mass 9464,300,
Subtract 102,037;

The available part of the stone = ... 9362,263.
The cubic inches to each therefore must be 1872,4526.

After the first man has ground off his share from the whole mass,
there must remain .... 7591,848 cubic inches;
After two have ground, there remains 5716,396 " ;
After three have ground, ... 3846,944 " ;
After four have ground, ... 1974,492 " ;
After the fifth, there remains ..... 102,050 " ,
which is the waste space about the axis.

The main part of the problem is now to come; that is, to discover how much each one will diminish the diameter. One general expression, however, will answer for the whole. Suppose the first man grinds down from A to EF. FH will then be the semidiameter. Conceive the plane CDEF to revolve on the axis CD : it will generate a solid = 7591,848 cubic inches, as we have just determined.

But this solid will consist of a cylinder and two equal cones. The length of the cylinder is GH, and the altitude of one of the cones is HD.

Let FH = y, and HD = x.

Then $$AO : OD = y : x;$$

that is, $$\frac{93}{2} : \frac{93}{44{,}5} = y : x :$$

hence $$x = \frac{4}{89}y.$$

The expression for the solidity of the cylinder is

$$2\tilde{\eta}y^2\left(\frac{93}{44{,}5} - x\right).$$

For the two cones, the expression is $\frac{2}{3}\tilde{\eta}y^2x.$

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Therefore the expression for the whole solid must be

$$\frac{186\tilde{\eta}y^2}{44,5} - 2\tilde{\eta}y^2x + \tfrac{2}{3}\tilde{\eta}y^2x, \qquad \text{or} \qquad \frac{186\tilde{\eta}y^2}{44,5} - \tfrac{4}{3}\tilde{\eta}y^2x.$$

Substituting the value of x in terms of y, we have the following general expression, which, being put equal to the cubic inches which remain after the first, second, third, or fourth person has ground off his share, will give an equation which, being solved, will give y, and of course $2y$, the diameter of the stone at each of these points, thus :

$$\frac{186\tilde{\eta}y^2}{44,5} - \frac{16\tilde{\eta}y^3}{267} = 7591,848.$$

This equation will give the diameter of the stone after the first man has ground off his share.

The above equation reduces to

$$y^3 - \frac{93,267}{89,4}y^2 = -(7591,848)\frac{267}{16\tilde{\eta}},$$

or

$$y^3 - 69,75y^2 = -(7591,848)\frac{267}{50,265}. \quad \text{------- [1]}$$

The following equation will give the diameter of the stone, after *two shares* are ground off :

$$y^3 - 69,75y^2 = -(5719,396)\frac{267}{50,265}; \quad \text{------- [2]}$$

the following, when *three shares* are ground off :

$$y^3 - 69,75y^2 = -(3846,944)\frac{267}{50,265}; \quad \text{------- [3]}$$

and the following, when *four shares* are taken :

$$y^3 - 69,75y^2 = -(1974,492)\frac{267}{50,265}. \quad \text{------- [4]}$$

These equations, being of the third degree, may have three real roots; and the mere mechanical operator might happen on one that would not apply to this problem, and from thence condemn the equations; but these circumstances will never trouble the initiated.

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Solving equation [1], we find *three roots;* one near 56, another 33,19 nearly, and a third near the locality of *minus* 21. As neither the first nor the third can apply to our problem, we take no pains to determine them more accurately. The second root, 33,19 inches nearly, shows that the first share taken off would diminish the radius of the stone 13,3 inches, or the diameter 26,6 inches.

Equation [2] has three roots; one near 62, the second 26,48 nearly, and the third near *minus* 19 : the second applies to our problem.

Equation [3] gives three roots; the first near 63,5, the second 22,77, and the third near *minus* 16,5.

Equation [4] also gives three roots; one near 65, the second 13,67, and the third near *minus* 12.

The fifth man will grind down to the eye. The radius of the circle, or y, will then be 2,828, and should be found in the following equation :

$$y^3 - 69{,}75y^2 = -(102{,}037)\frac{267}{50{,}265} = -541{,}8.$$

A solution of this equation gives $y = 2{,}845$; showing a slight discrepancy, probably arising from the neglect of decimals.

Whence the answer to the question is as follows :

The number of cubic inches in the stone is 9362,263.
The share for each of the five men is .... 1872,452.
The first share takes 26,6 inches from the diameter;
the second, 13,42 inches " ;
the third, 7,42 " " ;
the fourth, 18,20 " " ;
the fifth, 21,66 " " .

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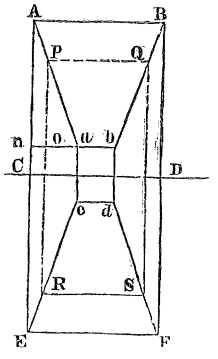
PROBLEM II.

There is a grindstone 6 *feet in diameter, with a six-inch circular eye-hole at the centre. The inner rim of the stone at the eye is but* 2 *inches in thickness, and outer rim* 6 *inches. What is the solid content of this stone, and what will be its diameter when one-third worn away?*



Let the plane ABab revolve on the axis CD, and conceive it to generate the stone.

ABCD, revolving on CD, will generate a cylinder, which will just enclose the stone, two conical frustums, and the cylindrical eye-space.

Therefore if from the cylinder generated by the plane ACDB, we subtract the two frustums and the small cylinder whose section is abcd, we shall have the solidity of the stone.

The solidity of the cylinder is found by the following product:

$0{,}7854(72)^2 6 = 24429{,}08$ cub. inches.

The frustums $\frac{4}{3}(\overline{72}^2+72.6+6^2)0{,}7854 = 5918{,}774$ "

$18510{,}30$ "

The small cylinder $0{,}7854(36)2$ .... $= 56{,}55$ "

Cubical content of the stone in inches $= 18453{,}75$ "

One-third of this .... $= 6151{,}25$ "

When the stone is worn down one-third, let its outer rim be at PQ; then PR will represent its diameter.

Now it is obvious that the cylinder generated by the revolution of the plane PQRS on the axis, together with *the two frustums at its ends*, must equal the whole original solid, diminished by 6151,25 cubic inches; that is, equal .. 24429,08

less 6151,25

which is equal to 18277,83

Draw ao perpendicular to PR and AC. Put ao $= x$, and

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